Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 42 (1996)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: CENTRALISERS IN THE BRAID GROUP AND SINGULAR BRAID MONOID
Autor: Fenn, Roger / Zhu, Jun
Kapitel: 7. Centralisers in SB_n
DOI: https://doi.org/10.5169/seals-87872

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 13.03.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

7. Centralisers in SB_n

7.1 Theorem. *For a singular braid $x \in SB_n$ the following are equivalent:*

(a) $\sigma_j x = x\sigma_k$;

(b) $\sigma_j^r x = x\sigma_k^r$, *for some nonzero integer* r;

(c) $\sigma_j^r x = x\sigma_k^r$, *for every integer* r;

(d) $\tau_j x = x\tau_k$;

(e) $\tau_j^r x = x\tau_k^r$, *for some nonzero integer* r;

(f) x *has a (possibly singular)* (j, k)*-band.*

Proof. We only show (b) implies (f). The other implications are quite clear. The term "band" will include the possibility of a singular band. Suppose $\sigma_j^r x = x\sigma_k^r$. Then we have that $\sigma_j^{2r} x = x\sigma_k^{2r}$. Assume $x = \beta\tau_i y$, where β is a braid; in other words τ_i is the first singular generator appearing in x. Then we have $\beta^{-1}\sigma_j^{2r}\beta\tau_i y = \tau_i y\sigma_k^{2r}$. Recall that isotopy, or the extended Reidemeister moves for singular braids, do not change the order of singular generators on the same strings. Since $\beta^{-1}\sigma_j^{2r}\beta$ is a pure braid, the τ_i in $\tau_i y\sigma_k^{2r}$ corresponds under some homeomorphism, to the τ_i in $\beta^{-1}\sigma_j^{2r}\beta\tau_i y$. Hence the image, under that homeomorphism, of the trivial singular band near the first τ_i provides a band for $\beta^{-1}\sigma_j^{2r}\beta$. Therefore, τ_i commutes with $\beta^{-1}\sigma_j^{2r}\beta$. It follows that $\tau_i\beta^{-1}\sigma_j^{2r}\beta y = \tau_i y\sigma_k^{2r}$. By Proposition 5.1, we have $\beta^{-1}\sigma_j^{2r}\beta y = y\sigma_k^{2r}$, i.e. $\sigma_j^{2r}\beta y = \beta y\sigma_k^{2r}$. By induction, βy has a (j, k)-band. Since τ_i commutes with $\beta^{-1}\sigma_j^{2r}\beta$, so does σ_i, thus we have $\beta^{-1}\sigma_j^{2r}\beta\sigma_i y = \sigma_i\beta^{-1}\sigma_j^{2r}\beta y = \sigma_i y\sigma_k^{2r}$, i.e. $\sigma_j^{2r}\beta\sigma_i y = \beta\sigma_i y\sigma_k^{2r}$. It follows from induction assumption that $\beta\sigma_i y$ has a (j, k)-band. Since both βy and $\beta\sigma_i y$ have a (j, k)-band, we can use the argument of Lemma 6.4 to conclude that $x = \beta\tau_i y$ has a (j, k)-band. □

The above theorem allows us to identify monoid centralisers in SB_n. Notice that SB_2 is abelian. On the other hand, for $n \geqslant 3$, any singular braid with a singularity involving strings labelled, say, j and k, $j < k$, could not possibly commute with τ_k, as any (singular) band from $[k, k+1] \times 0$ to $[k, k+1] \times 1$ would have a forbidden intersection with the j string, see Figure 7. Therefore for $n \geqslant 3$, only *nonsingular* braids are central. We will conclude with two applications whose proofs, at this point, can safely be left to the reader.

7.2 THEOREM. *The centre of* SB_n *is all of* SB_n *for* $n = 2$. *But in case* $n \geqslant 3$ *it is the same as the (infinite cyclic) centre of* $B_n \subset SB_n$, *generated by* Δ^2. □

7.3 THEOREM. *Under the natural inclusion, the centraliser of* SB_r *in* SB_n, $r \leqslant n$, *is generated as a monoid by the generators (see Theorem 4.4) of* $C(r, n)$:

$$\sigma_{r+1}, \sigma_{r+2}, \ldots, \sigma_{n-1}, A_{r+1}, \ldots, A_n, C,$$

together with their inverses and the singular generators:

$$\tau_{r+1}, \ldots, \tau_{n-1} \quad \text{if } r \geqslant 3, \text{ or}$$
$$\tau_1, \tau_3, \tau_4, \ldots, \tau_{n-1} \quad \text{if } r = 2.$$ □

REFERENCES

[Art 1] ARTIN, E. Theorie der Zöpfe. *Abh. Math. Sem. Hamburg Univ. 4* (1926), 47-72.

[Art 2] —— Theory of Braids. *Ann. of Math. (2) 48* (1947), 101-128.

[Bae] BAEZ, J. Link invariants and perturbation theory. *Lett. Math. Phys. 2* (1992), 43-51.

[Bar] BAR-NATAN, D. On the Vassiliev knot invariants. *Topology 34* (1995), 423-471.

[Bir1] BIRMAN, J. *Braids, links and mapping class groups.* Ann. Math. Studies 82, Princeton University Press, Princeton, N.J., 1974.

[Bir2] —— New points of view in knot theory. *Bull. AMS 28* (1993), 253-287.

[Bur] BURDE, G. Über Normalisatoren der Zopfgruppe. *Abh. Math. Sem. Hamburg 27* (1964), 97-115.

[BZ] BURDE, G. and H. ZIESCHANG. *Knots.* De Gruyter Studies in Math. 5, New York, 1985.

[Char1] CHARNEY, R. Artin groups of finite type are biautomatic. *Math. Ann. 292* (1992), 671-693.

[Char2] —— Geodesic automation and growth functions for Artin groups of finite type. *Math. Ann. 301* (1995), 307-324.

[Chow] CHOW, W.-L. On the algebraical braid group. *Ann. of Math. (2) 49* (1948), 654-658.

[ECHLPT] EPSTEIN, D., J. CANNON, D. HOLT, S. LEVY, M. PATERSON and W. THURSTON. *Word processing in groups.* Jones and Bartlett, 1992.

[FRR] FENN, R., R. RIMÁNYI and C. ROURKE. The braid-permutation group. Preprint (submitted for publication).

[Gar] GARSIDE, F. The braid group and other groups. *Quart. J. Math. Oxford Ser. (2) 20* (1969), 235-254.

[GS] GERSTEN, S. and H. SHORT. Rational subgroups of biautomatic groups. *Ann. Math. 105* (1991), 641-662.