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Artikel: CENTRALISERS IN THE BRAID GROUP AND SINGULAR BRAID MONOID

Autor: Fenn, Roger / Zhu, Jun

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5.2 PROPOSITION. *Suppose* $x \in SB_n^{(t)}$ *is a singular braid of degree* s. *Then* $\eta(x) \in \mathbf{Z}B_n$ *is a linear combination of* 2^t *elements of* B_n *(call them terms). There is a unique term of maximal degree* $s+t$ *and a unique term of minimal degree* $s-t$. *More generally, for each integer* $u, 0 \leqslant u \leqslant t, \eta(x)$ *has* $\binom{t}{u}$ *terms of degree* $s+t-2u$, *and each of these terms has coefficient* $(-1)^u$. $\square$

There may be some cancellation among the terms of degree strictly between $s-t$ and $s+t$, but since there is only one term of maximum and one term of minimal degree, they cannot be cancelled and we draw the following conclusions.

5.3 COROLLARY. *No element of* SB_n *maps to zero under* η. $\square$

The kernel of η is also trivial in another sense.

5.4 COROLLARY. *If* $1 \in B_n \subset SB_n$ *denotes the identity braid, then* $\eta^{-1}(1) = 1$. $\square$

To close this section we consider the natural extension of η to the monoid ring $\mathbf{Z}SB_n$.

5.5 PROPOSITION. *The extension* $\eta : \mathbf{Z}SB_n \to \mathbf{Z}B_n$ *is not injective.*

Proof. τ_1 and $\sigma_1 - \sigma_1^{-1}$ are two elements of $\mathbf{Z}SB_n$ with the same image. For a more subtle example, consider the elements

$$x = \tau_1\tau_2\sigma_1^{-1} + \tau_1\sigma_2\tau_1, \quad y = \tau_2\sigma_1^{-1}\tau_2 + \sigma_2\tau_1\tau_2 .$$

An easy calculation verifies that $\eta(x) = \eta(y)$. However, $x \neq y$, as can be seen by examining their images under the map $\tau_i \to \sigma_i, \sigma_i \to \sigma_i$. $\square$

The above example is related to certain canonical relations obeyed by the Vassiliev invariants — see [Bir2], p. 274, or [Bar].

6. RESULTS REGARDING INJECTIVITY OF η

Note that if $x, y \in SB_n$ satisfy $\eta(x) = \eta(y)$, then they both have the same number of singularities, i.e. $x \in SB_n^{(t)}$ if and only if $y \in SB_n^{(t)}$. The relevance of bands to the injectivity question will be illustrated by first checking injectivity of η restricted to $SB_n^{(1)}$. (Of course, it is injective on $SB_n^{(0)} = B_n$, because it is simply the inclusion of the basis of $\mathbf{Z}B_n$.)

6.1 LEMMA. *For a braid* $\beta \in B_n$, *the following are equivalent:*

(a) $\tau_i\beta = \beta\tau_j$,

(b) $\tau_i^m\beta = \beta\tau_j^m$ *for some positive integer* m.

(c) β *has an* (i, j)*-band.*

Proof. Clearly (a) $\Rightarrow$ (b) and, using the homomorphism $SB_n \to B_n$ defined by $\tau_k \to \sigma_k$, $\sigma_k \to \sigma_k$, we see that (b) implies $\sigma_i^m\beta = \beta\sigma_j^m$, which implies (c) by Theorem 2.2. Finally, (c) $\Rightarrow$ (a), because the band can be used to convey τ_i on the left of β to become τ_j on the right. $\square$

In Section 7 we will prove a generalisation of this lemma in which β is allowed to be a singular braid.

6.2 THEOREM. *If* $x, y \in SB_n^{(1)}$ *and* $\eta(x) = \eta(y)$, *then* $x = y$.

Proof. We can write $x = \alpha\tau_i\beta$ and $y = \alpha'\tau_j\beta'$ for (nonsingular) braids $\alpha, \alpha', \beta, \beta'$ and compute:

$$\eta(x) = \alpha\sigma_i\beta - \alpha\sigma_i^{-1}\beta\ ,$$
$$\eta(y) = \alpha'\sigma_j\beta' - \alpha'\sigma_j^{-1}\beta'\ .$$

Equating the terms of highest and lowest degree, we have:

$$\alpha\sigma_i\beta = \alpha'\sigma_j\beta' \quad \text{and} \quad \alpha\sigma_i^{-1}\beta = \alpha'\sigma_j^{-1}\beta'\ .$$

It follows that

$$\sigma_i^2(\beta\beta'^{-1}) = (\beta\beta'^{-1})\sigma_j^2$$

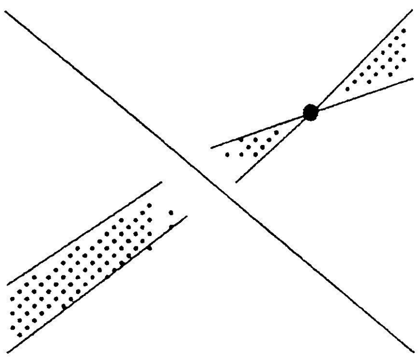
and, by the lemma,

$$\tau_i(\beta\beta'^{-1}) = (\beta\beta'^{-1})\tau_j,$$
$$\sigma_i(\beta\beta'^{-1}) = (\beta\beta'^{-1})\sigma_j\ .$$

We quickly deduce that $\beta\beta'^{-1} = \alpha\alpha'^{-1}$ and it follows that

$$\alpha\tau_i\beta = \alpha'\tau_j\beta' \qquad \square$$

We will now work towards the injectivity of η on $SB_n^{(2)}$. Define a *singular ribbon* to be a map $R: \mathbf{I} \times \mathbf{I} \to \mathbf{C} \times \mathbf{I}$ such that R embeds $\mathbf{I} \times t$ into $\mathbf{C} \times t$, except for finitely many points t, for which the image is a single point in $\mathbf{C} \times t$. One also assumes, at these singular points, that there is a tangent plane in $\mathbf{C} \times \mathbf{I}$ for the singular ribbon. Singular ribbons are the best one can do for ribbons for singular braids. As with braids, we say a singular ribbon is *proper* for a singular braid if it sends $\{0, 1\} \times \mathbf{I}$ along two of its strands and the image is disjoint from the other strings of the singular braid. An isotopy of a singular braid can be extended to an isotopy of any of its proper singular ribbons, with the following caveat: under the equivalence $\tau_i\tau_j = \tau_j\tau_i$ one may have to reparametrise the singular ribbon.



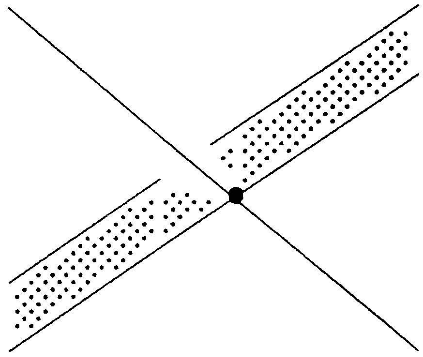


FIGURE 7

Singular ribbons only intersect two strands of a singular braid

In contrast to the situation for ordinary braids, it is not always possible to find a singular ribbon proper for a given singular braid x and with a given arc A as its intersection with $\mathbf{C} \times 0$. For example, consider an $(i, i+1)$-arc A, suppose β is a braid such that $\{i, i+1\} * \beta = \{j, j+1\}$ and consider a singular braid x of the form $x = \beta\tau_j \cdots$. Then a necessary condition for the existence of a singular ribbon, whose intersection with $\mathbf{C} \times 0$ is A, would be $A * \beta = [j, j+1]$. On the other hand, for the same reason as for ribbons, we do have the following.

6.3 PROPOSITION. *If a singular ribbon R is proper for the singular braid x and $R(\mathbf{I} \times 0)$ and $R(\mathbf{I} \times 1)$ are isotopic as proper arcs to $[j, j+1] \times 0$ and $[k, k+1] \times 1$, respectively, then $\sigma_i x = x\sigma_j$ in SB_n.* $\square$

DEFINITION. We will extend our previous definition and say that a singular braid *has a (j, k)-band* if it has a proper ribbon or singular ribbon connecting $[j, j+1] \times 0$ to $[k, k+1] \times 1$. The crucial facts we've proved are that a braid β has a (j, k)-band if and only if $\sigma_j\beta = \beta\sigma_k$, and for singular braids, having a (j, k)-band is a sufficient condition for satisfying such an equation.

6.4 LEMMA. *Let* α, β *be braids such that both* $\alpha\sigma_i\beta$ *and* $\alpha\beta$ *have* (j,k)*-bands. Then* $\alpha\tau_i\beta$ *also has a* (j,k)*-band.*

Proof. Consideration of the induced permutation implies that the pair $\{j, j+1\} * \alpha$ is either $\{i, i+1\}$ (case 1) or disjoint from $\{i, i+1\}$ (case 2). In either case, let $A = [j, j+1] * \alpha$. Then, since $\alpha\beta$ has a (j,k)-band we have $[j, j+1] * (\alpha\beta) = [k, k+1]$, and so $A = [k, k+1] * \beta^{-1} = \beta * [k, k+1]$. Similarly the hypothesis that $\alpha\sigma_i\beta$ has a (j,k)-band implies that $A * \sigma_i = A$.

Now, in case 1, A is an $(i, i+1)$-arc and we must have $A * \sigma_i = \bar{A}$. Lemma 3.2 implies that $A = [i, i+1]$. We conclude that α has a (j,i)-band and β has an (i,k)-band, and these combine with the obvious singular (i,i)-band for τ_i to provide a (j,k)-band for $\alpha\tau_i\beta$.

In case 2, Lemma 3.1 applies, and we may assume after an isotopy of the (j,k) band for $\alpha\beta$ that its intersection, A, with $C \times 1/2$ is disjoint from $[i, i+1]$. This implies that we may insert τ_i between α and β so that the singular strands are disjoint from the band, and we conclude that $\alpha\tau_i\beta$ has a nonsingular (j,k)-band. $\square$

6.5 THEOREM. *The map* η *is injective on* $SB_n^{(2)}$.

Proof. Consider an equation of the form

$$\eta(\alpha\tau_i\beta\tau_j\gamma) = \eta(\alpha'\tau_{i'}\beta'\tau_{j'}\gamma')$$

where $\alpha, \alpha', \beta, \beta', \gamma, \gamma', \in B_n$.

Now

$$\eta(\alpha\tau_i\beta\tau_j\gamma) = \alpha\sigma_i\beta\sigma_j\gamma - \alpha\sigma_i^{-1}\beta\sigma_j\gamma - \alpha\sigma_i\beta\sigma_j^{-1}\gamma + \alpha\sigma_i^{-1}\beta\sigma_j^{-1}\gamma$$

and $\eta(\alpha'\tau_{i'}\beta'\tau_{j'}\gamma')$ has a similar expansion. If they are equal in $\mathbf{Z}B_n$, then considering the degrees we must have one of two sets of equations. Either

$$\alpha\sigma_i\beta\sigma_j\gamma = \alpha'\sigma_{i'}\beta'\sigma_{j'}\gamma' \tag{1}$$

$$\alpha\sigma_i^{-1}\beta\sigma_j\gamma = \alpha'\sigma_{i'}^{-1}\beta'\sigma_{j'}\gamma' \tag{2}$$

$$\alpha\sigma_i\beta\sigma_j^{-1}\gamma = \alpha'\sigma_{i'}\beta'\sigma_{j'}^{-1}\gamma' \tag{3}$$

$$\alpha\sigma_i^{-1}\beta\sigma_j^{-1}\gamma = \alpha'\sigma_{i'}^{-1}\beta'\sigma_{j'}^{-1}\gamma' \tag{4}$$

or

$$\alpha\sigma_i\beta\sigma_j\gamma = \alpha'\sigma_{i'}\beta'\sigma_{j'}\gamma' \tag{1}$$

$$\alpha\sigma_i^{-1}\beta\sigma_j\gamma = \alpha'\sigma_{i'}\beta'\sigma_{j'}^{-1}\gamma' \tag{2'}$$

$$\alpha\sigma_i\beta\sigma_j^{-1}\gamma = \alpha'\sigma_{i'}^{-1}\beta'\sigma_{j'}\gamma' \tag{3'}$$

$$\alpha\sigma_i^{-1}\beta\sigma_j^{-1}\gamma = \alpha'\sigma_{i'}^{-1}\beta'\sigma_{j'}^{-1}\gamma' \tag{4}$$

We claim that in either case the following are true:

(5) $$\alpha\beta\gamma = \alpha'\beta'\gamma'$$

(6) $$\alpha\tau_i\beta\tau_j\gamma = \alpha'\tau_{i'}\beta'\tau_{j'}\gamma' .$$

Assume initially that (1), (2), (3) and (4) are satisfied. Eliminating $\beta'\sigma_{j'}\gamma'$ between (1) and (2) gives $\alpha'^{-1}\alpha\sigma_i^2 = \sigma_{i'}^2\alpha'^{-1}\alpha$. The main theorem now implies that $\alpha'^{-1}\alpha$ has an (i', i)-band. Similarly eliminating $\alpha'\sigma_{i'}\beta'$ between (1) and (3) implies that $\gamma\gamma'^{-1}$ has a (j, j')-band. Applying these facts to (1) gives

$$\sigma_{i'}\beta'\sigma_{j'} = \alpha'^{-1}\alpha\sigma_i\beta\sigma_j\gamma\gamma'^{-1} = \sigma_{i'}\alpha'^{-1}\alpha\beta\gamma\gamma'^{-1}\sigma_{j'}$$

and (5) follows in this case.

Similarly using (5)

$$\tau_{i'}\beta'\tau_{j'} = \tau_{i'}\alpha'^{-1}\alpha\beta\gamma\gamma'^{-1}\tau_j = \alpha'^{-1}\alpha\tau_i\beta\tau_j\gamma\gamma'^{-1} ,$$

and therefore (6) also holds in this case.

Now assume that the equations (1), (2′), (3′) and (4) hold. A similar elimination as in the first case implies that $\beta\sigma_j\gamma\gamma'^{-1}$ has an (i, j')-band and $\alpha'^{-1}\alpha\sigma_i\beta$ has an (i', j)-band. So

$$\sigma_{i'}\beta'\sigma_{j'} = \alpha'^{-1}\alpha\sigma_i\beta\sigma_j\gamma\gamma'^{-1} = \sigma_{i'}\alpha'^{-1}\alpha\sigma_i\beta\gamma\gamma'^{-1}$$

The above can be written as

(7) $$\alpha\sigma_i\beta\gamma = \alpha'\beta'\sigma_{j'}\gamma'$$

Similarly from equation (4) we have

(8) $$\alpha\sigma_i^{-1}\beta\gamma = \alpha'\beta'\sigma_{j'}^{-1}\gamma'$$

Eliminating $\alpha^{-1}\alpha'\beta'$ between (7) and (8) gives $\sigma_i^2\beta\gamma\gamma'^{-1} = \beta\gamma\gamma'^{-1}\sigma_{j'}^2$, so $\beta\gamma\gamma'^{-1}$ has an (i, j')-band, and with Lemma 6.6 we deduce that $\beta\tau_j\gamma\gamma'^{-1}$ has an (i, j')-band. We can also conclude that equation (5) holds in this case. A similar argument shows that $\alpha'^{-1}\alpha\beta$ has an (i', j)-band.

Hence

$$\begin{aligned}
\alpha'^{-1}\alpha\tau_i\beta\tau_j\gamma\gamma'^{-1} &= \alpha'^{-1}\alpha\beta\tau_j\gamma\gamma'^{-1}\tau_{j'} && (i, j')\text{-band}\\
&= \tau_{i'}\alpha'^{-1}\alpha\beta\gamma\gamma'^{-1}\tau_{j'} && (i', j)\text{-band}\\
&= \tau_{i'}\beta'\tau_{j'} .
\end{aligned}$$

So (6) is true in this case as well. □