

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 42 (1996)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** CENTRALISERS IN THE BRAID GROUP AND SINGULAR BRAID MONOID  
**Autor:** Fenn, Roger / Zhu, Jun  
**Kapitel:** 6. Results regarding injectivity of  
**DOI:** <https://doi.org/10.5169/seals-87872>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 19.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

**5.2 PROPOSITION.** *Suppose  $x \in SB_n^{(t)}$  is a singular braid of degree  $s$ . Then  $\eta(x) \in \mathbb{Z}B_n$  is a linear combination of  $2^t$  elements of  $B_n$  (call them terms). There is a unique term of maximal degree  $s + t$  and a unique term of minimal degree  $s - t$ . More generally, for each integer  $u$ ,  $0 \leq u \leq t$ ,  $\eta(x)$  has  $\binom{t}{u}$  terms of degree  $s + t - 2u$ , and each of these terms has coefficient  $(-1)^u$ .  $\square$*

There may be some cancellation among the terms of degree strictly between  $s - t$  and  $s + t$ , but since there is only one term of maximum and one term of minimal degree, they cannot be cancelled and we draw the following conclusions.

**5.3 COROLLARY.** *No element of  $SB_n$  maps to zero under  $\eta$ .  $\square$*

The kernel of  $\eta$  is also trivial in another sense.

**5.4 COROLLARY.** *If  $1 \in B_n \subset SB_n$  denotes the identity braid, then  $\eta^{-1}(1) = 1$ .  $\square$*

To close this section we consider the natural extension of  $\eta$  to the monoid ring  $\mathbb{Z}SB_n$ .

**5.5 PROPOSITION.** *The extension  $\eta: \mathbb{Z}SB_n \rightarrow \mathbb{Z}B_n$  is not injective.*

*Proof.*  $\tau_1$  and  $\sigma_1 - \sigma_1^{-1}$  are two elements of  $\mathbb{Z}SB_n$  with the same image. For a more subtle example, consider the elements

$$x = \tau_1 \tau_2 \sigma_1^{-1} + \tau_1 \sigma_2 \tau_1, \quad y = \tau_2 \sigma_1^{-1} \tau_2 + \sigma_2 \tau_1 \tau_2.$$

An easy calculation verifies that  $\eta(x) = \eta(y)$ . However,  $x \neq y$ , as can be seen by examining their images under the map  $\tau_i \rightarrow \sigma_i, \sigma_i \rightarrow \sigma_i$ .  $\square$

The above example is related to certain canonical relations obeyed by the Vassiliev invariants — see [Bir2], p. 274, or [Bar].

## 6. RESULTS REGARDING INJECTIVITY OF $\eta$

Note that if  $x, y \in SB_n$  satisfy  $\eta(x) = \eta(y)$ , then they both have the same number of singularities, i.e.  $x \in SB_n^{(t)}$  if and only if  $y \in SB_n^{(t)}$ . The relevance of bands to the injectivity question will be illustrated by first checking

injectivity of  $\eta$  restricted to  $SB_n^{(1)}$ . (Of course, it is injective on  $SB_n^{(0)} = B_n$ , because it is simply the inclusion of the basis of  $\mathbb{Z}B_n$ .)

6.1 LEMMA. *For a braid  $\beta \in B_n$ , the following are equivalent:*

- (a)  $\tau_i \beta = \beta \tau_j$ ,
- (b)  $\tau_i^m \beta = \beta \tau_j^m$  for some positive integer  $m$ .
- (c)  $\beta$  has an  $(i, j)$ -band.

*Proof.* Clearly (a)  $\Rightarrow$  (b) and, using the homomorphism  $SB_n \rightarrow B_n$  defined by  $\tau_k \rightarrow \sigma_k$ ,  $\sigma_k \rightarrow \sigma_k$ , we see that (b) implies  $\sigma_i^m \beta = \beta \sigma_j^m$ , which implies (c) by Theorem 2.2. Finally, (c)  $\Rightarrow$  (a), because the band can be used to convey  $\tau_i$  on the left of  $\beta$  to become  $\tau_j$  on the right.  $\square$

In Section 7 we will prove a generalisation of this lemma in which  $\beta$  is allowed to be a singular braid.

6.2 THEOREM. *If  $x, y \in SB_n^{(1)}$  and  $\eta(x) = \eta(y)$ , then  $x = y$ .*

*Proof.* We can write  $x = \alpha \tau_i \beta$  and  $y = \alpha' \tau_j \beta'$  for (nonsingular) braids  $\alpha, \alpha', \beta, \beta'$  and compute:

$$\begin{aligned}\eta(x) &= \alpha \sigma_i \beta - \alpha \sigma_i^{-1} \beta, \\ \eta(y) &= \alpha' \sigma_j \beta' - \alpha' \sigma_j^{-1} \beta' .\end{aligned}$$

Equating the terms of highest and lowest degree, we have:

$$\alpha \sigma_i \beta = \alpha' \sigma_j \beta' \quad \text{and} \quad \alpha \sigma_i^{-1} \beta = \alpha' \sigma_j^{-1} \beta' .$$

It follows that

$$\sigma_i^2 (\beta \beta'^{-1}) = (\beta \beta'^{-1}) \sigma_j^2$$

and, by the lemma,

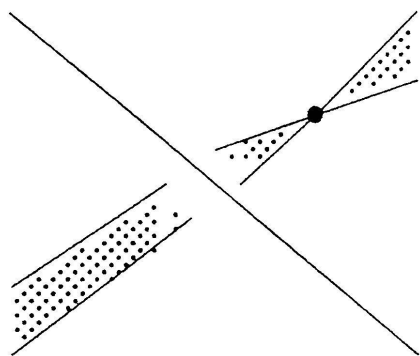
$$\begin{aligned}\tau_i (\beta \beta'^{-1}) &= (\beta \beta'^{-1}) \tau_j, \\ \sigma_i (\beta \beta'^{-1}) &= (\beta \beta'^{-1}) \sigma_j .\end{aligned}$$

We quickly deduce that  $\beta \beta'^{-1} = \alpha \alpha'^{-1}$  and it follows that

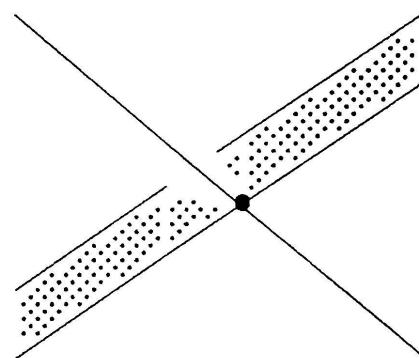
$$\alpha \tau_i \beta = \alpha' \tau_j \beta' \quad \square$$

We will now work towards the injectivity of  $\eta$  on  $SB_n^{(2)}$ . Define a *singular ribbon* to be a map  $R: \mathbf{I} \times \mathbf{I} \rightarrow \mathbf{C} \times \mathbf{I}$  such that  $R$  embeds  $\mathbf{I} \times t$  into  $\mathbf{C} \times t$ , except for finitely many points  $t$ , for which the image is a single point in  $\mathbf{C} \times t$ . One also assumes, at these singular points, that there is a

tangent plane in  $\mathbf{C} \times \mathbf{I}$  for the singular ribbon. Singular ribbons are the best one can do for ribbons for singular braids. As with braids, we say a singular ribbon is *proper* for a singular braid if it sends  $\{0, 1\} \times \mathbf{I}$  along two of its strands and the image is disjoint from the other strings of the singular braid. An isotopy of a singular braid can be extended to an isotopy of any of its proper singular ribbons, with the following caveat: under the equivalence  $\tau_i \tau_j = \tau_j \tau_i$  one may have to reparametrise the singular ribbon.



A singular ribbon



NOT a singular ribbon

FIGURE 7

Singular ribbons only intersect two strands of a singular braid

In contrast to the situation for ordinary braids, it is not always possible to find a singular ribbon proper for a given singular braid  $x$  and with a given arc  $A$  as its intersection with  $\mathbf{C} \times 0$ . For example, consider an  $(i, i+1)$ -arc  $A$ , suppose  $\beta$  is a braid such that  $\{i, i+1\} * \beta = \{j, j+1\}$  and consider a singular braid  $x$  of the form  $x = \beta \tau_j \cdots$ . Then a necessary condition for the existence of a singular ribbon, whose intersection with  $\mathbf{C} \times 0$  is  $A$ , would be  $A * \beta = [j, j+1]$ . On the other hand, for the same reason as for ribbons, we do have the following.

**6.3 PROPOSITION.** *If a singular ribbon  $R$  is proper for the singular braid  $x$  and  $R(\mathbf{I} \times 0)$  and  $R(\mathbf{I} \times 1)$  are isotopic as proper arcs to  $[j, j+1] \times 0$  and  $[k, k+1] \times 1$ , respectively, then  $\sigma_i x = x \sigma_j$  in  $SB_n$ .  $\square$*

**DEFINITION.** We will extend our previous definition and say that a singular braid *has a  $(j, k)$ -band* if it has a proper ribbon or singular ribbon connecting  $[j, j+1] \times 0$  to  $[k, k+1] \times 1$ . The crucial facts we've proved are that a braid  $\beta$  has a  $(j, k)$ -band if and only if  $\sigma_j \beta = \beta \sigma_k$ , and for singular braids, having a  $(j, k)$ -band is a sufficient condition for satisfying such an equation.

6.4 LEMMA. *Let  $\alpha, \beta$  be braids such that both  $\alpha\sigma_i\beta$  and  $\alpha\beta$  have  $(j, k)$ -bands. Then  $\alpha\tau_i\beta$  also has a  $(j, k)$ -band.*

*Proof.* Consideration of the induced permutation implies that the pair  $\{j, j+1\} * \alpha$  is either  $\{i, i+1\}$  (case 1) or disjoint from  $\{i, i+1\}$  (case 2). In either case, let  $A = [j, j+1] * \alpha$ . Then, since  $\alpha\beta$  has a  $(j, k)$ -band we have  $[j, j+1] * (\alpha\beta) = [k, k+1]$ , and so  $A = [k, k+1] * \beta^{-1} = \beta * [k, k+1]$ . Similarly the hypothesis that  $\alpha\sigma_i\beta$  has a  $(j, k)$ -band implies that  $A * \sigma_i = A$ .

Now, in case 1,  $A$  is an  $(i, i+1)$ -arc and we must have  $A * \sigma_i = \bar{A}$ . Lemma 3.2 implies that  $A = [i, i+1]$ . We conclude that  $\alpha$  has a  $(j, i)$ -band and  $\beta$  has an  $(i, k)$ -band, and these combine with the obvious singular  $(i, i)$ -band for  $\tau_i$  to provide a  $(j, k)$ -band for  $\alpha\tau_i\beta$ .

In case 2, Lemma 3.1 applies, and we may assume after an isotopy of the  $(j, k)$  band for  $\alpha\beta$  that its intersection,  $A$ , with  $C \times 1/2$  is disjoint from  $[i, i+1]$ . This implies that we may insert  $\tau_i$  between  $\alpha$  and  $\beta$  so that the singular strands are disjoint from the band, and we conclude that  $\alpha\tau_i\beta$  has a nonsingular  $(j, k)$ -band.  $\square$

6.5 THEOREM. *The map  $\eta$  is injective on  $SB_n^{(2)}$ .*

*Proof.* Consider an equation of the form

$$\eta(\alpha\tau_i\beta\tau_j\gamma) = \eta(\alpha'\tau_{i'}\beta'\tau_{j'}\gamma')$$

where  $\alpha, \alpha', \beta, \beta', \gamma, \gamma' \in B_n$ .

Now

$$\eta(\alpha\tau_i\beta\tau_j\gamma) = \alpha\sigma_i\beta\sigma_j\gamma - \alpha\sigma_i^{-1}\beta\sigma_j\gamma - \alpha\sigma_i\beta\sigma_j^{-1}\gamma + \alpha\sigma_i^{-1}\beta\sigma_j^{-1}\gamma$$

and  $\eta(\alpha'\tau_{i'}\beta'\tau_{j'}\gamma')$  has a similar expansion. If they are equal in  $\mathbf{Z}B_n$ , then considering the degrees we must have one of two sets of equations. Either

$$(1) \quad \alpha\sigma_i\beta\sigma_j\gamma = \alpha'\sigma_{i'}\beta'\sigma_{j'}\gamma'$$

$$(2) \quad \alpha\sigma_i^{-1}\beta\sigma_j\gamma = \alpha'\sigma_{i'}^{-1}\beta'\sigma_{j'}\gamma'$$

$$(3) \quad \alpha\sigma_i\beta\sigma_j^{-1}\gamma = \alpha'\sigma_{i'}\beta'\sigma_{j'}^{-1}\gamma'$$

$$(4) \quad \alpha\sigma_i^{-1}\beta\sigma_j^{-1}\gamma = \alpha'\sigma_{i'}^{-1}\beta'\sigma_{j'}^{-1}\gamma'$$

or

$$(1) \quad \alpha\sigma_i\beta\sigma_j\gamma = \alpha'\sigma_{i'}\beta'\sigma_{j'}\gamma'$$

$$(2') \quad \alpha\sigma_i^{-1}\beta\sigma_j\gamma = \alpha'\sigma_{i'}\beta'\sigma_{j'}^{-1}\gamma'$$

$$(3') \quad \alpha\sigma_i\beta\sigma_j^{-1}\gamma = \alpha'\sigma_{i'}^{-1}\beta'\sigma_{j'}\gamma'$$

$$(4) \quad \alpha\sigma_i^{-1}\beta\sigma_j^{-1}\gamma = \alpha'\sigma_{i'}^{-1}\beta'\sigma_{j'}^{-1}\gamma'$$

We claim that in either case the following are true:

$$(5) \quad \alpha\beta\gamma = \alpha'\beta'\gamma'$$

$$(6) \quad \alpha\tau_i\beta\tau_j\gamma = \alpha'\tau_{i'}\beta'\tau_{j'}\gamma'.$$

Assume initially that (1), (2), (3) and (4) are satisfied. Eliminating  $\beta'\sigma_{j'}\gamma'$  between (1) and (2) gives  $\alpha'^{-1}\alpha\sigma_i^2 = \sigma_{i'}^2\alpha'^{-1}\alpha$ . The main theorem now implies that  $\alpha'^{-1}\alpha$  has an  $(i', i)$ -band. Similarly eliminating  $\alpha'\sigma_{i'}\beta'$  between (1) and (3) implies that  $\gamma\gamma'^{-1}$  has a  $(j, j')$ -band. Applying these facts to (1) gives

$$\sigma_{i'}\beta'\sigma_{j'} = \alpha'^{-1}\alpha\sigma_i\beta\sigma_j\gamma\gamma'^{-1} = \sigma_{i'}\alpha'^{-1}\alpha\beta\gamma\gamma'^{-1}\sigma_{j'}$$

and (5) follows in this case.

Similarly using (5)

$$\tau_{i'}\beta'\tau_{j'} = \tau_{i'}\alpha'^{-1}\alpha\beta\gamma\gamma'^{-1}\tau_j = \alpha'^{-1}\alpha\tau_i\beta\tau_j\gamma\gamma'^{-1},$$

and therefore (6) also holds in this case.

Now assume that the equations (1), (2'), (3') and (4) hold. A similar elimination as in the first case implies that  $\beta\sigma_j\gamma\gamma'^{-1}$  has an  $(i, j')$ -band and  $\alpha'^{-1}\alpha\sigma_i\beta$  has an  $(i', j)$ -band. So

$$\sigma_{i'}\beta'\sigma_{j'} = \alpha'^{-1}\alpha\sigma_i\beta\sigma_j\gamma\gamma'^{-1} = \sigma_{i'}\alpha'^{-1}\alpha\sigma_i\beta\gamma\gamma'^{-1}$$

The above can be written as

$$(7) \quad \alpha\sigma_i\beta\gamma = \alpha'\beta'\sigma_{j'}\gamma'$$

Similarly from equation (4) we have

$$(8) \quad \alpha\sigma_i^{-1}\beta\gamma = \alpha'\beta'\sigma_{j'}^{-1}\gamma'$$

Eliminating  $\alpha^{-1}\alpha'\beta'$  between (7) and (8) gives  $\sigma_i^2\beta\gamma\gamma'^{-1} = \beta\gamma\gamma'^{-1}\sigma_{j'}^2$ , so  $\beta\gamma\gamma'^{-1}$  has an  $(i, j')$ -band, and with Lemma 6.6 we deduce that  $\beta\tau_j\gamma\gamma'^{-1}$  has an  $(i, j')$ -band. We can also conclude that equation (5) holds in this case. A similar argument shows that  $\alpha'^{-1}\alpha\beta$  has an  $(i', j)$ -band.

Hence

$$\begin{aligned} \alpha'^{-1}\alpha\tau_i\beta\tau_j\gamma\gamma'^{-1} &= \alpha'^{-1}\alpha\beta\tau_j\gamma\gamma'^{-1}\tau_{j'} && (i, j')\text{-band} \\ &= \tau_{i'}\alpha'^{-1}\alpha\beta\gamma\gamma'^{-1}\tau_{j'} && (i', j)\text{-band} \\ &= \tau_{i'}\beta'\tau_{j'}. \end{aligned}$$

So (6) is true in this case as well.  $\square$