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1-2: L'ENSEIGNEMENT MATHÉMATIQUE
CENTRALISERS IN THE BRAID GROUP AND SINGULAR BRAID MONOID
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3. Proof of Theorem 2.2
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2.4 COROLLARY. The inner automorphism in  $B_n$  exchanging generators,  $\sigma_k = \beta^{-1} \sigma_j \beta$ , is achieved exactly by those braids  $\beta$  that have a (j, k)-band.  $\Box$ 

2.5 COROLLARY [Chow]. The centre of  $B_n$ ,  $n \ge 3$  is infinite cyclic, generated by the braid  $\Delta^2$ , where

$$\Delta = \sigma_{n-1}(\sigma_{n-2}\sigma_{n-1})\cdots(\sigma_1\sigma_2\cdots\sigma_{n-1}).$$

**Proof.** A braid commutes with all braid generators if and only if its action stabilises all the intervals [1, 2], ..., [n - 1, n], so it has a great ribbon containing the entire braid, connecting  $[1, n] \times 0$  with  $[1, n] \times 1$ , necessarily in an order-preserving sense. Such a braid is clearly a multiple of the full-twist  $\Delta^2$ .

# 3. PROOF OF THEOREM 2.2

It is useful here to introduce an invariant of proper arcs. Throughout this section A will denote an oriented (k, l)-arc in C which is proper with respect to  $\{1, ..., n\}$ .

Associated with A is a word in the symbols  $I_0, I_1, ..., I_n, I_0^{-1}$ ,  $I_1^{-1}, ..., I_n^{-1}$  which can be described as follows. Assume that A is transverse to the real line. Starting from its initial point k, continue along A to l and whenever A crosses the interval [m, m + 1] write  $I_m$  if it crosses with increasing imaginary part and write  $I_m^{-1}$  otherwise. In the above notation, use the interval  $(-\infty, 1]$  in case m = 0 and  $[n, \infty)$  if m = n, in place of [m, m + 1]. An isotopy of A will change the word by a sequence of moves of the following sort:

- a) the introduction or deletion of cancelling pairs of the form  $I_m I_m^{-1}$  or  $I_m^{-1} I_m$ ,
- b) left multiplication by a word in  $I_{k-1}$ ,  $I_k$  and
- c) right multiplication by a word in  $I_{l-1}, I_l$ .

Let w(A) be the word in the free group on the symbols  $I_0, I_1, ..., I_n$ obtained by deleting all cancelling pairs, all initial segments in  $I_{k-1}, I_k$  and all final segments in  $I_{l-1}, I_l$ . Then w(A) is an isotopy invariant, and it is routine to check that A can be isotoped to read off exactly the word w(A). Note that the exponents  $\pm 1$  of symbols in w(A) necessarily alternate. The action of  $\sigma_j$  on the word w(A) is as follows, in the case that the ends of A are not in the set  $\{j, j + 1\}$ :

$$I_{m}^{\pm 1} \to I_{m}^{\pm 1} \qquad \text{if} \quad m \neq j ,$$
  

$$I_{j} \to I_{j-1} I_{j}^{-1} I_{j+1} ,$$
  

$$I_{j}^{-1} \to I_{j+1}^{-1} I_{j} I_{j-1}^{-1} .$$

If an end of A happens to be j - 1 or j + 2, one may also have to delete an initial or final  $I_{j-1}^{\pm 1}$  or  $I_{j+1}^{\pm 1}$ , after applying the above transformation.

Although not needed in our proof of Theorem 2.2, the next lemma will be useful later.

3.1 LEMMA. If A is a (k, l)-arc, with  $\{k, l\} \cap \{j, j+1\} = \emptyset$ , such that  $A * \sigma_j = A$ , then up to isotopy A is disjoint from [j, j+1].

**Proof.** It suffices to show that w(A), if reduced, does not contain  $I_j^{\pm 1}$ . It follows from the above rules that each occurrence of  $I_j$  in w(A) is replaced by exactly one occurrence with opposite sign in  $w(A * \sigma_j)$ , and if we are to have  $w(A) = w(A * \sigma_j)$  there will be no cancellations among the  $I_j$  in  $w(A * \sigma_j)$ . So if  $I_j$  occurs, we conclude  $w(A) \neq w(A * \sigma_j)$ , contradicting  $A * \sigma_j = A$ .

3.2 LEMMA. If A is a (j, j + 1)-arc such that  $A * \sigma_j^r = A$  for some integer  $r \neq 0$ , then up to isotopy A = [j, j + 1].

*Proof.* Noting that  $A * \sigma_j^r = A$  if and only if  $A * \sigma_j^{-r} = A$ , we assume, without loss of generality, that r > 0. By iteration we have  $A * \sigma_j^{2r} = A$ . The lemma will follow if we can show that w(A) must reduce to the empty word. So we suppose (for contradiction) that w(A) is nonempty. First, note that then w(A) must involve some symbol  $I_p$  with  $|p - j| \ge 2$ . (For otherwise  $A \in \mathbb{C} - \{(-\infty, j - 1] \cup [j + 2, +\infty)\}$ , which is homeomorphic with  $\mathbb{C}$  itself; but it is well-known that any two arcs in  $\mathbb{C}$  are isotopic with fixed ends, and we would have A isotopic to [j, j + 1] and w(A) empty.)

We assume the first and last symbols of w(A) have exponent +1 (the other three cases can be argued similarly, or follow by symmetry). Then, referring to Figure 4, we have:

$$w(A * \sigma_j^{2r}) = (I_{j+1}I_{j-1}^{-1})^r w^* (I_{j+1}^{-1}I_{j-1})^{-r}$$

where  $w^*$  is the transformation of w(A) according to the rules (\*) above,

iterated 2r times. Noting that  $I_p$  persists in  $w^*$  it is easy to argue that  $w(A * \sigma_j^{2r}) = w(A)$  is impossible; the contradiction.



FIGURE 4 The action of  $*\sigma_j^{2r}$  on a (j, k)-arc in case r = 2

We now turn to the proof of Theorem 2.2. It has already been observed that  $(e) \Rightarrow (d) \Rightarrow (a)$ , and it is obvious that  $(a) \Rightarrow (c) \Rightarrow (b)$ . So it remains to establish that  $(b) \Rightarrow (e)$ . Thus we assume that, for some  $r \neq 0$ ,  $\sigma_j^r \beta = \beta \sigma_k^r$ . Since the algebraic crossing number of any two strings of a braid is a well-defined braid invariant, this equation is possible only if  $\{j, j + 1\} * \beta$  $= \{k, k + 1\}$ . Now, noting that  $\beta^{-1}\sigma_j^r \beta = \sigma_k^r$  and that  $\sigma_k^r$  has a (k, k)-band, we conclude that there is a proper ribbon for  $\beta^{-1}\sigma_j^r \beta$  from  $[k, k + 1] \times 0$ to  $[k, k + 1] \times 1$ . Define  $A = \beta * [k, k + 1] = [k, k + 1] * \beta^{-1}$ . Then we may assume (possibly after an isotopy) that the planes  $\mathbb{C} \times 1/3$  and  $\mathbb{C} \times 2/3$  cut the ribbon in the arcs  $A \times 1/3$  and  $A \times 2/3$ . Moreover, the middle third of the ribbon, and Proposition 1.1, imply that  $A * \sigma_j^r = A$ . By Lemma 3.2, A = [j, j + 1] and the theorem is proved.

## 4. CENTRALISERS OF BRAID SUBGROUPS

We have established the following.

4.1 THEOREM. The centraliser in  $B_n$  of the generator  $\sigma_j$  is the subgroup of all braids which have (j, j)-bands. This subgroup is isomorphic to  $B_{n-1}^j \times \mathbb{Z}$  where  $B_{n-1}^j$  is the subgroup of  $B_{n-1}$  consisting of all (n-1)-braids whose permutations stabilise j.

The goal of this section is to describe the centraliser of  $B_r$  in  $B_n$ ,  $r \leq n$ , which we will call C(r, n). Here  $B_r$  is the r-string braid group with its usual inclusion in  $B_n$ , namely as the subgroup generated by  $\sigma_1 \dots \sigma_{r-1}$ .