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**Autor:** Fenn, Roger / Zhu, Jun  
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2.4 COROLLARY. *The inner automorphism in  $B_n$  exchanging generators,  $\sigma_k = \beta^{-1} \sigma_j \beta$ , is achieved exactly by those braids  $\beta$  that have a  $(j, k)$ -band.  $\square$*

2.5 COROLLARY [Chow]. *The centre of  $B_n$ ,  $n \geq 3$  is infinite cyclic, generated by the braid  $\Delta^2$ , where*

$$\Delta = \sigma_{n-1}(\sigma_{n-2}\sigma_{n-1}) \cdots (\sigma_1\sigma_2 \cdots \sigma_{n-1}).$$

*Proof.* A braid commutes with all braid generators if and only if its action stabilises all the intervals  $[1, 2], \dots, [n-1, n]$ , so it has a great ribbon containing the entire braid, connecting  $[1, n] \times 0$  with  $[1, n] \times 1$ , necessarily in an order-preserving sense. Such a braid is clearly a multiple of the full-twist  $\Delta^2$ .  $\square$

### 3. PROOF OF THEOREM 2.2

It is useful here to introduce an invariant of proper arcs. Throughout this section  $A$  will denote an oriented  $(k, l)$ -arc in  $\mathbb{C}$  which is proper with respect to  $\{1, \dots, n\}$ .

Associated with  $A$  is a word in the symbols  $I_0, I_1, \dots, I_n, I_0^{-1}, I_1^{-1}, \dots, I_n^{-1}$  which can be described as follows. Assume that  $A$  is transverse to the real line. Starting from its initial point  $k$ , continue along  $A$  to  $l$  and whenever  $A$  crosses the interval  $[m, m+1]$  write  $I_m$  if it crosses with increasing imaginary part and write  $I_m^{-1}$  otherwise. In the above notation, use the interval  $(-\infty, 1]$  in case  $m = 0$  and  $[n, \infty)$  if  $m = n$ , in place of  $[m, m+1]$ . An isotopy of  $A$  will change the word by a sequence of moves of the following sort:

- a) the introduction or deletion of cancelling pairs of the form  $I_m I_m^{-1}$  or  $I_m^{-1} I_m$ ,
- b) left multiplication by a word in  $I_{k-1}, I_k$  and
- c) right multiplication by a word in  $I_{l-1}, I_l$ .

Let  $w(A)$  be the word in the free group on the symbols  $I_0, I_1, \dots, I_n$  obtained by deleting all cancelling pairs, all initial segments in  $I_{k-1}, I_k$  and all final segments in  $I_{l-1}, I_l$ . Then  $w(A)$  is an isotopy invariant, and it is routine to check that  $A$  can be isotoped to read off exactly the word  $w(A)$ . Note that the exponents  $\pm 1$  of symbols in  $w(A)$  necessarily alternate.

The action of  $\sigma_j$  on the word  $w(A)$  is as follows, in the case that the ends of  $A$  are not in the set  $\{j, j+1\}$ :

$$\begin{aligned} I_m^{\pm 1} &\rightarrow I_m^{\pm 1} && \text{if } m \neq j, \\ I_j &\rightarrow I_{j-1} I_j^{-1} I_{j+1}, \\ I_j^{-1} &\rightarrow I_{j+1}^{-1} I_j I_{j-1}^{-1}. \end{aligned}$$

If an end of  $A$  happens to be  $j-1$  or  $j+2$ , one may also have to delete an initial or final  $I_{j-1}^{\pm 1}$  or  $I_{j+1}^{\pm 1}$ , after applying the above transformation.

Although not needed in our proof of Theorem 2.2, the next lemma will be useful later.

**3.1 LEMMA.** *If  $A$  is a  $(k, l)$ -arc, with  $\{k, l\} \cap \{j, j+1\} = \emptyset$ , such that  $A * \sigma_j = A$ , then up to isotopy  $A$  is disjoint from  $[j, j+1]$ .*

*Proof.* It suffices to show that  $w(A)$ , if reduced, does not contain  $I_j^{\pm 1}$ . It follows from the above rules that each occurrence of  $I_j$  in  $w(A)$  is replaced by exactly one occurrence with opposite sign in  $w(A * \sigma_j)$ , and if we are to have  $w(A) = w(A * \sigma_j)$  there will be no cancellations among the  $I_j$  in  $w(A * \sigma_j)$ . So if  $I_j$  occurs, we conclude  $w(A) \neq w(A * \sigma_j)$ , contradicting  $A * \sigma_j = A$ .  $\square$

**3.2 LEMMA.** *If  $A$  is a  $(j, j+1)$ -arc such that  $A * \sigma_j^r = A$  for some integer  $r \neq 0$ , then up to isotopy  $A = [j, j+1]$ .*

*Proof.* Noting that  $A * \sigma_j^r = A$  if and only if  $A * \sigma_j^{-r} = A$ , we assume, without loss of generality, that  $r > 0$ . By iteration we have  $A * \sigma_j^{2r} = A$ . The lemma will follow if we can show that  $w(A)$  must reduce to the empty word. So we suppose (for contradiction) that  $w(A)$  is nonempty. First, note that then  $w(A)$  must involve some symbol  $I_p$  with  $|p - j| \geq 2$ . (For otherwise  $A \subset \mathbf{C} - \{(-\infty, j-1] \cup [j+2, +\infty)\}$ , which is homeomorphic with  $\mathbf{C}$  itself; but it is well-known that any two arcs in  $\mathbf{C}$  are isotopic with fixed ends, and we would have  $A$  isotopic to  $[j, j+1]$  and  $w(A)$  empty.)

We assume the first and last symbols of  $w(A)$  have exponent  $+1$  (the other three cases can be argued similarly, or follow by symmetry). Then, referring to Figure 4, we have:

$$w(A * \sigma_j^{2r}) = (I_{j+1} I_{j-1}^{-1})^r w^* (I_{j+1}^{-1} I_{j-1})^{-r}$$

where  $w^*$  is the transformation of  $w(A)$  according to the rules (\*) above,

iterated  $2r$  times. Noting that  $I_p$  persists in  $w^*$  it is easy to argue that  $w(A * \sigma_j^{2r}) = w(A)$  is impossible; the contradiction.  $\square$

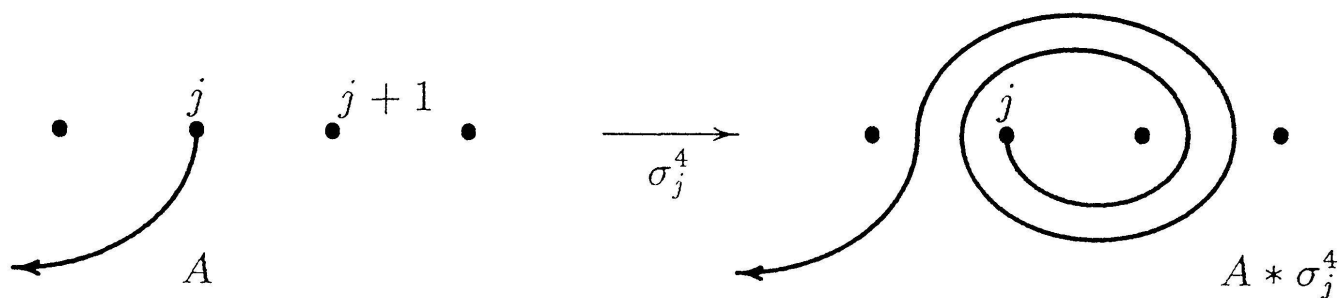


FIGURE 4

The action of  $*\sigma_j^{2r}$  on a  $(j, k)$ -arc in case  $r = 2$

We now turn to the proof of Theorem 2.2. It has already been observed that  $(e) \Rightarrow (d) \Rightarrow (a)$ , and it is obvious that  $(a) \Rightarrow (c) \Rightarrow (b)$ . So it remains to establish that  $(b) \Rightarrow (e)$ . Thus we assume that, for some  $r \neq 0$ ,  $\sigma_j^r \beta = \beta \sigma_k^r$ . Since the algebraic crossing number of any two strings of a braid is a well-defined braid invariant, this equation is possible only if  $\{j, j+1\} * \beta = \{k, k+1\}$ . Now, noting that  $\beta^{-1} \sigma_j^r \beta = \sigma_k^r$  and that  $\sigma_k^r$  has a  $(k, k)$ -band, we conclude that there is a proper ribbon for  $\beta^{-1} \sigma_j^r \beta$  from  $[k, k+1] \times 0$  to  $[k, k+1] \times 1$ . Define  $A = \beta * [k, k+1] = [k, k+1] * \beta^{-1}$ . Then we may assume (possibly after an isotopy) that the planes  $\mathbf{C} \times 1/3$  and  $\mathbf{C} \times 2/3$  cut the ribbon in the arcs  $A \times 1/3$  and  $A \times 2/3$ . Moreover, the middle third of the ribbon, and Proposition 1.1, imply that  $A * \sigma_j^r = A$ . By Lemma 3.2,  $A = [j, j+1]$  and the theorem is proved.  $\square$

#### 4. CENTRALISERS OF BRAID SUBGROUPS

We have established the following.

**4.1 THEOREM.** *The centraliser in  $B_n$  of the generator  $\sigma_j$  is the subgroup of all braids which have  $(j, j)$ -bands. This subgroup is isomorphic to  $B_{n-1}^j \times \mathbf{Z}$  where  $B_{n-1}^j$  is the subgroup of  $B_{n-1}$  consisting of all  $(n-1)$ -braids whose permutations stabilise  $j$ .  $\square$*

The goal of this section is to describe the centraliser of  $B_r$  in  $B_n$ ,  $r \leq n$ , which we will call  $C(r, n)$ . Here  $B_r$  is the  $r$ -string braid group with its usual inclusion in  $B_n$ , namely as the subgroup generated by  $\sigma_1 \dots \sigma_{r-1}$ .