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Artikel: CENTRALISERS IN THE BRAID GROUP AND SINGULAR BRAID MONOID

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2.4 COROLLARY. *The inner automorphism in* B_n *exchanging generators,* $\sigma_k = \beta^{-1}\sigma_j\beta$, *is achieved exactly by those braids* β *that have a* (j, k)*-band.* □

2.5 COROLLARY [Chow]. *The centre of* B_n, $n \geqslant 3$ *is infinite cyclic, generated by the braid* Δ^2, *where*

$$\Delta = \sigma_{n-1}(\sigma_{n-2}\sigma_{n-1}) \cdots (\sigma_1\sigma_2 \cdots \sigma_{n-1}) .$$

Proof. A braid commutes with all braid generators if and only if its action stabilises all the intervals $[1, 2], \ldots, [n-1, n]$, so it has a great ribbon containing the entire braid, connecting $[1, n] \times 0$ with $[1, n] \times 1$, necessarily in an order-preserving sense. Such a braid is clearly a multiple of the full-twist Δ^2. □

3. PROOF OF THEOREM 2.2

It is useful here to introduce an invariant of proper arcs. Throughout this section A will denote an oriented (k, l)-arc in $\mathbf{C}$ which is proper with respect to $\{1, \ldots, n\}$.

Associated with A is a word in the symbols $I_0, I_1, \ldots, I_n, I_0^{-1}, I_1^{-1}, \ldots, I_n^{-1}$ which can be described as follows. Assume that A is transverse to the real line. Starting from its initial point k, continue along A to l and whenever A crosses the interval $[m, m+1]$ write I_m if it crosses with increasing imaginary part and write I_m^{-1} otherwise. In the above notation, use the interval $(-\infty, 1]$ in case $m = 0$ and $[n, \infty)$ if $m = n$, in place of $[m, m+1]$. An isotopy of A will change the word by a sequence of moves of the following sort:

a) the introduction or deletion of cancelling pairs of the form $I_m I_m^{-1}$ or $I_m^{-1} I_m$,

b) left multiplication by a word in I_{k-1}, I_k and

c) right multiplication by a word in I_{l-1}, I_l.

Let $w(A)$ be the word in the free group on the symbols $I_0, I_1, \ldots, I_n$ obtained by deleting all cancelling pairs, all initial segments in I_{k-1}, I_k and all final segments in I_{l-1}, I_l. Then $w(A)$ is an isotopy invariant, and it is routine to check that A can be isotoped to read off exactly the word $w(A)$. Note that the exponents ± 1 of symbols in $w(A)$ necessarily alternate.

The action of σ_j on the word $w(A)$ is as follows, in the case that the ends of A are not in the set $\{j, j+1\}$:

$$\begin{aligned} I_m^{\pm 1} &\to I_m^{\pm 1} \qquad\qquad \text{if} \quad m \neq j\,, \\ I_j &\to I_{j-1} I_j^{-1} I_{j+1}\,, \\ I_j^{-1} &\to I_{j+1}^{-1} I_j I_{j-1}^{-1}\,. \end{aligned}$$

If an end of A happens to be $j-1$ or $j+2$, one may also have to delete an initial or final $I_{j-1}^{\pm 1}$ or $I_{j+1}^{\pm 1}$, after applying the above transformation.

Although not needed in our proof of Theorem 2.2, the next lemma will be useful later.

3.1 LEMMA. *If A is a (k, l)-arc, with $\{k, l\} \cap \{j, j+1\} = \emptyset$, such that $A * \sigma_j = A$, then up to isotopy A is disjoint from $[j, j+1]$.*

Proof. It suffices to show that $w(A)$, if reduced, does not contain $I_j^{\pm 1}$. It follows from the above rules that each occurrence of I_j in $w(A)$ is replaced by exactly one occurrence with opposite sign in $w(A * \sigma_j)$, and if we are to have $w(A) = w(A * \sigma_j)$ there will be no cancellations among the I_j in $w(A * \sigma_j)$. So if I_j occurs, we conclude $w(A) \neq w(A * \sigma_j)$, contradicting $A * \sigma_j = A$. □

3.2 LEMMA. *If A is a $(j, j+1)$-arc such that $A * \sigma_j^r = A$ for some integer $r \neq 0$, then up to isotopy $A = [j, j+1]$.*

Proof. Noting that $A * \sigma_j^r = A$ if and only if $A * \sigma_j^{-r} = A$, we assume, without loss of generality, that $r > 0$. By iteration we have $A * \sigma_j^{2r} = A$. The lemma will follow if we can show that $w(A)$ must reduce to the empty word. So we suppose (for contradiction) that $w(A)$ is nonempty. First, note that then $w(A)$ must involve some symbol I_p with $|p - j| \geqslant 2$. (For otherwise $A \subset \mathbf{C} - \{(-\infty, j-1] \cup [j+2, +\infty)\}$, which is homeomorphic with $\mathbf{C}$ itself; but it is well-known that any two arcs in $\mathbf{C}$ are isotopic with fixed ends, and we would have A isotopic to $[j, j+1]$ and $w(A)$ empty.)

We assume the first and last symbols of $w(A)$ have exponent $+1$ (the other three cases can be argued similarly, or follow by symmetry). Then, referring to Figure 4, we have:

$$w(A * \sigma_j^{2r}) = (I_{j+1} I_{j-1}^{-1})^r w^* (I_{j+1}^{-1} I_{j-1})^{-r}$$

where w^* is the transformation of $w(A)$ according to the rules $(*)$ above, iterated $2r$ times. Noting that I_p persists in w^* it is easy to argue that $w(A * \sigma_j^{2r}) = w(A)$ is impossible; the contradiction. $\square$

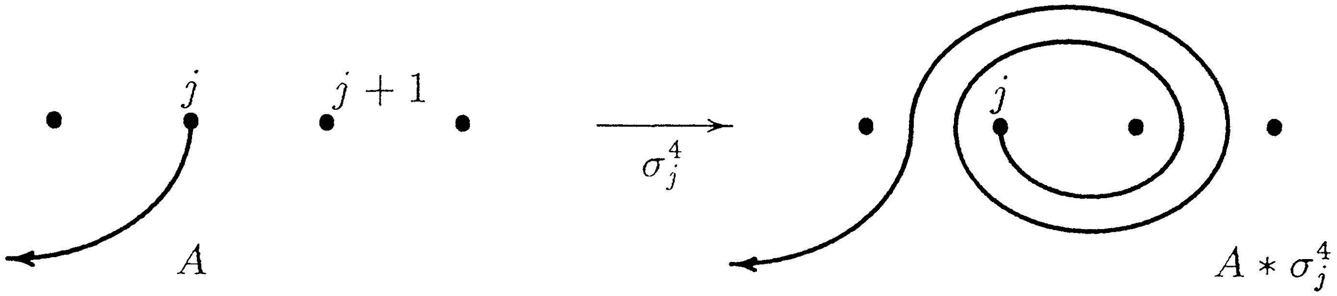

FIGURE 4
The action of $*\sigma_j^{2r}$ on a (j, k)-arc in case $r = 2$

We now turn to the proof of Theorem 2.2. It has already been observed that $(e) \Rightarrow (d) \Rightarrow (a)$, and it is obvious that $(a) \Rightarrow (c) \Rightarrow (b)$. So it remains to establish that $(b) \Rightarrow (e)$. Thus we assume that, for some $r \neq 0$, $\sigma_j^r \beta = \beta \sigma_k^r$. Since the algebraic crossing number of any two strings of a braid is a well-defined braid invariant, this equation is possible only if $\{j, j+1\} * \beta = \{k, k+1\}$. Now, noting that $\beta^{-1} \sigma_j^r \beta = \sigma_k^r$ and that σ_k^r has a (k, k)-band, we conclude that there is a proper ribbon for $\beta^{-1}\sigma_j^r\beta$ from $[k, k+1] \times 0$ to $[k, k+1] \times 1$. Define $A = \beta * [k, k+1] = [k, k+1] * \beta^{-1}$. Then we may assume (possibly after an isotopy) that the planes $\mathbf{C} \times 1/3$ and $\mathbf{C} \times 2/3$ cut the ribbon in the arcs $A \times 1/3$ and $A \times 2/3$. Moreover, the middle third of the ribbon, and Proposition 1.1, imply that $A * \sigma_j^r = A$. By Lemma 3.2, $A = [j, j+1]$ and the theorem is proved. $\square$

4. CENTRALISERS OF BRAID SUBGROUPS

We have established the following.

4.1 THEOREM. *The centraliser in B_n of the generator σ_j is the subgroup of all braids which have (j, j)-bands. This subgroup is isomorphic to $B_{n-1}^j \times \mathbf{Z}$ where B_{n-1}^j is the subgroup of B_{n-1} consisting of all $(n-1)$-braids whose permutations stabilise j.* $\square$

The goal of this section is to describe the centraliser of B_r in B_n, $r \leqslant n$, which we will call $C(r, n)$. Here B_r is the r-string braid group with its usual inclusion in B_n, namely as the subgroup generated by $\sigma_1 \ldots \sigma_{r-1}$.