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OVER TORSION-FREE GROUPS
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notation. Let r_i be the road common to the boundary of D and the country with stopping car κ_i . Let the end junctions of r_i be v_i and v_{i+1} in anticlockwise order $i = 1, 2, \dots$. Suppose κ_i is on the road r_i . Then by hypothesis the roads r_{i+1} and r_{i-1} are free of the cars κ_{i+1} and κ_{i-1} respectively (see figure 3).

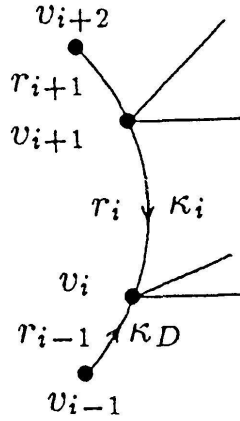


FIGURE 3

The motion of car κ_D

As κ_i traverses r_i from v_{i+1} to v_i let κ_D traverse r_{i-1} from v_{i-1} to v_i . Let the cars meet at v_i at time t . This will not be a complete crash since κ_{i-1} is missing. Again by hypothesis κ_{i+1} will not be at v_{i+2} at time t . Let r be largest such that κ_{i+r} is not at v_{i+r+1} at time t . Then κ_D has enough time to arrive at v_{i+r+1} just as κ_{i+r} does. If there is no such r then let κ_{i+r} be the next car to arrive at D and let κ_D go once round the entire boundary and arrive at v_{i+r+1} just as κ_{i+r} does. Keep repeating this strategem to define the motion of κ_D .

Now we are in a situation corresponding to the first crash theorem and the result is proved. \square

3. TWO TRANSVERSALITY LEMMAS

In this section we use transversality (cf. [BRS, F]) to prove the existence of diagrams of van-Kampen type for the two situations that we shall meet in the applications to group theory of the crash theorems (in sections 4, 5 and 6). These lemmas need to be stated very carefully and a failure to do so is one of the major weaknesses in Klyachko's version. The lemmas use the idea of a *corner* of a 2-cell in a cell subdivision K of the 2-sphere. This can be regarded as the (oriented) angle formed by the two adjacent edges meeting at a 0-cell in the boundary of the 2-cell. If all the corners of a 2-cell are labelled by elements of a group, then a word can be read around the 2-cell

boundary by composing these elements either unchanged or inverted according as the orientation of the corner agrees or disagrees with that of the 2-cell boundary. Similarly if all the corners at a 0-cell are labelled then a word can be read around that 0-cell. We shall always orient corners *clockwise*, thus if the above words are read *clockwise* for 0-cells and *anticlockwise* for 2-cells, then no inversion is necessary (see figure 4).



FIGURE 4

Multiplying the corner labels to get $g_1 g_2 \cdots g_n$ for a 0-cell and a 2-cell

Let $w \in A * B$ be an element in the free product of two groups. We shall only be interested in w up to cyclic reordering and thus (cyclically reordering if necessary) we can assume that either $w = 1$ or $w \in A \cup B - 1$ or w is written uniquely as $a_1 b_1 a_2 b_2 \cdots a_n b_n$ where a_i and b_i are non-trivial elements of A, B alternately. These non-trivial elements a_i, b_i are then called the (cyclic) *factors* of w .

LEMMA 3.1. *Let A, B be two groups and let $N = \langle\langle W \rangle\rangle$ be the normal closure in $A * B$ of some subset of elements $W \subset A * B$. Suppose $N \cap A \neq \{1\}$. Then there is a cell subdivision of the sphere S^2 such that each corner of each 2-cell is labelled by an element of $A \cup B$ with the following properties.*

1. *The corner labels of a 2-cell are the cyclic factors (in anticlockwise order and up to cyclic rotation) of some w or w^{-1} where $w \in W$.*
2. *The corner labels at a 0-cell are either all in A or all in B .*
3. *The (clockwise) product of the corner labels at a 0-cell is 1 (in A or B) except for one special 0-cell where the product is a non-trivial element of $A \cup B$.*

Proof. Let K_A, K_B be two disjoint 2-dimensional complexes such that $\pi_1(K_A, *_A) = A$ and $\pi_1(K_B, *_B) = B$. Join the base points $*_A$ and $*_B$ by an arc α with central point $*$. Let $K = K_A \cup \alpha \cup K_B$. Then $\pi_1(K, *)$

$\cong A * B$. Attach 2-cells σ_w to K by the words $w \in W$ to form the complex L . If $a \in N \cap A - \{1\}$ there is a map $f: D^2, S^1 \rightarrow L, K$ from the 2-disc to L such that the restriction $f|S^1$ to the boundary represents a . Make the map f transverse to the centres of the 2-cells σ_w . It follows that the inverse images of small neighbourhoods of these centres is a collection of disjoint discs D_1, \dots, D_m in the interior of D^2 . By a radial expansion of f on these discs we may assume that each image is the whole of one of the σ_w . It follows that the punctured disc $P = D^2 - \overline{D_1 \cup \dots \cup D_m}$ is mapped by f to K . Make $f|P$ transverse to $*$. Then $f^{-1}*$ is a 1-manifold Z properly embedded in P . By a radial expansion along α we can assume that Z has a neighbourhood N which is a normal I -bundle and where each fibre is mapped by f to α . The complementary space $P - N$ is divided into connected regions which are mapped by f to K_A or K_B . On crossing N one passes from one kind of region to the other.

We now simplify the subset $D_1 \cup \dots \cup D_m \cup N$ of D^2 as follows. Suppose N contains an annulus component \mathcal{A} in the interior of P . Let D' denote the interior disc of D^2 which bounds the interior boundary component of the annulus. Then $D' \cup \mathcal{A}$ is a sub disc of D^2 whose boundary gets mapped to a base point by f . We can then shrink it to a point, redefine f and simplify the situation. Having eliminated all annuli, $D_1 \cup \dots \cup D_m \cup N$ will look like a thickened graph in D^2 with the discs D_i corresponding to thickened vertices and the components of N to thickened edges. Our next task is to make this graph connected. If not choose an innermost component C . Draw a simple loop around C separating it from the rest of $D_1 \cup \dots \cup D_m \cup N$. This loop will represent (up to conjugacy) an element of $A \cup B$. If this element is trivial we can shrink the disc it bounds as above and simplify the situation. If not we replace $D_1 \cup \dots \cup D_m \cup N$ by C . Note that the boundary curve may now represent a non trivial element of B instead of A .

Attach a 2-cell (outside) to the boundary of D^2 and label the centre of this outside cell ∞ . The 2-disc has now become a 2-sphere. In this situation consider the dual graph Γ . This has a vertex in each region and an edge joining neighbouring regions separated by a component of N . For the outer region take the vertex to be ∞ . Then Γ and its complementary regions define a cell subdivision K of the 2-sphere. Each vertex is either in an A region or a B region and the corners can be correspondingly labelled by elements of A or B as follows. Every 2-cell of K contains a unique subdisc D_i . Opposite a corner is an edge of D_i labelled by an element of A or B . Take this to be the labelling of the corner. By moving anticlockwise around the

boundary of a 2-cell of K the corner labellings spell out a cyclic rotation of some w_i or w_i^{-1} . By moving clockwise around a 0-cell of K the corner labellings spell out the trivial element (of A or B) except for ∞ which spells out a non-trivial element of A or B . \square

NOTE. It may not be possible to specify that the non-trivial element lies in A as this simple example shows. Let $A = \langle a \rangle$, $B = \langle b \rangle$ be two infinite cyclic groups generated by a, b respectively. Let the words of the attaching 2-cells be ab^{-1}, b . In this case the 2-cells of the required subdivision have either two corners (those modelled on ab^{-1}) or one corner (modelled on b) and the only possible subdivision of the 2-sphere satisfying lemma 3.1 is the trivial one with single vertex labelled b . This is a place where Klyachko's version is definitely wrong (rather than badly stated).

Let $w \in G * \langle t \rangle$ be an element of the free product of a group G with the infinite cyclic group $\langle t \rangle$. Then w can be written uniquely (up to cyclic rotation) in the form $w = g_1 t^{\varepsilon_1} g_2 \cdots t^{\varepsilon_n}$ where each $g_i \in G$, each $\varepsilon_i = \pm 1$ and g_i can only be 1 if it has neighbouring t 's (in cyclic order) with the same exponent. We call g_1, \dots, g_n the *coefficients* of w .

The following lemma is proved in [H₁]. It is closely related to "pictures" [R₁, R₂, Sh].

LEMMA 3.2. *Let G be a group and consider the free product $G * \langle t \rangle$ of G with an infinite cyclic group (generator t). Let $N = \langle\langle W \rangle\rangle$ be the normal closure in $G * \langle t \rangle$ of some subset of elements $W \subset G * \langle t \rangle$. Suppose $N \cap G \neq \{1\}$ then there is a cell subdivision K of the 2-sphere such that*

- a) *the 1-cells of K are oriented,*
- b) *the corners (all oriented clockwise) are labelled by coefficients of elements of W ,*
- c) *the clockwise product of the corner labelling around any 0-cell is 1 except for one vertex where it is non trivial,*
- d) *the corner labels of any 2-cell (in anticlockwise order) are the coefficients of w or w^{-1} for some $w \in W$ (up to cyclic rotation) with the property that, if on passing from one corner to an adjacent corner the element t or t^{-1} is inserted according to whether the intervening edge is oriented in the same or opposite direction, then the whole of w or w^{-1} is recovered.*

Proof. The proof is very similar to 3.1. Let K_G be a 2-dimensional complex such that $\pi_1(K_G, *_G) = G$. Adjoin an oriented 1-cell γ to the base point $*_G$ to form a 2-dimensional complex $K = K_G \vee S^1$ with $\pi_1 K = G * \langle t \rangle$. Attach 2-cells to K by the words $w \in W$ to form L . Since $N \cap G \neq \{1\}$ there is a non contractable loop in K_G represented by a map $f: S^1, 1 \rightarrow K_G, *_G$ which can be extended to a map $f: D^2 \rightarrow L$.

We now proceed as in the proof of lemma 3.1 with the rôle of $*$ played by a point p in the interior of γ . We construct a graph whose (thickened) vertices are the inverse image of the 2-cells and whose edges are the inverse image of p . By making similar simplifications and passing to an innermost component, as before, we may assume that this graph is connected. Replace D^2 by a sphere as before. The dual subdivision now defines K . The orientation of the 1-cells is determined by the orientation of γ and it only remains to observe that these oriented edges correspond to the new generator t . \square

4. APPLICATION TO THE KERVAIRE PROBLEM

In this section we give Klyachko's application of the crash theorems to prove theorem 1.1 in the case in which exponent sum of t in the word w is 1. As remarked in the introduction this implies the Kervaire conjecture for torsion-free groups.

We say that a system of equations $\{w(t) = 1 \mid w \in W\}$ in the variable t , with coefficients in a group G , has a *solution over G* if there is a group \tilde{G} containing G as a subgroup and an element $x \in \tilde{G}$ such that the relations $\{w(x) = 1 \mid w \in W\}$ are satisfied in \tilde{G} . It is clear that this is equivalent to the natural map

$$G \rightarrow \frac{G * \langle t \rangle}{\langle\langle W \rangle\rangle}$$

being injective, where $\langle\langle W \rangle\rangle$ denotes the normal closure of W in $G * \langle t \rangle$.

Now let H be a subgroup of G and let $g \in G$. We say that g is *free relative to H* if the subgroup $\langle g, H \rangle$ of G generated by g and H is naturally the free product $\langle g \rangle * H$ of an infinite cyclic group $\langle g \rangle$ with H .

We shall apply the crash theorem with stops to prove theorem 4.1 (below) and then use an algebraic trick to deduce the case $\text{ex}(w) = 1$ of theorem 1.1.

If g, h are elements of a group let g^h denote $h^{-1}gh$.