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KLYACHKO'S METHODS AND THE SOLUTION OF EQUATIONS OVER TORSION-FREE GROUPS

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SUMMARY. The question we are concerned with here is the following:

Let G be a torsion-free group and consider the free product $G * \langle t \rangle$ of G with an infinite cyclic group (generator t). Let w be an element of $G * \langle t \rangle - G$ and $\langle\langle w \rangle\rangle$ denote the normal closure of w in $G * \langle t \rangle$, then is the natural homomorphism

$$G \rightarrow \frac{G * \langle t \rangle}{\langle\langle w \rangle\rangle}$$

injective?

Klyachko's paper: "Funny property of sphere and equations over groups" [Kl] contains a proof that it is injective in the case in which the exponent sum of t in w is 1. If the exponent sum is not ± 1 then $\frac{G * \langle t \rangle}{\langle\langle w \rangle\rangle}$ has a non-trivial cyclic quotient. So the following is implied:

COROLLARY (Kervaire conjecture for torsion-free groups). *Let G be a non-trivial torsion-free group then $\frac{G * \langle t \rangle}{\langle\langle w \rangle\rangle}$ is non-trivial.*

The proof for exponent sum 1 is based on Klyachko's "funny property of sphere". This is the following: Let K be a cell subdivision of the 2-sphere with at least one 1-cell. Let a car drive round the boundary of each 2-cell in an anti-clockwise sense (the cars travel at arbitrary speeds, never stop and visit each point of the boundary of the cell infinitely often). Then there must be at least two places on the sphere where complete crashes occur (a complete crash is either a head-on collision in the middle of a 1-cell or a crash at a vertex involving all the cars from neighbouring 2-cells).

Klyachko describes this property as "suitable for a school mathematics tournament". The property is used to show that the diagram for a potential counterexample to the Kervaire conjecture must have at least one interior vertex with all labels being the same element of G, hence this element has finite order.

In this paper we shall give an exposition of Klyachko's methods and theorems. We use his techniques to give a positive answer to the question for other exponents under a technical condition on the t-shape of w, for details here see section 5.

1. INTRODUCTION

In [Kl] Klyachko gives a proof of the Kervaire conjecture in the situation where the groups involved are torsion-free. Unfortunately the paper suffers from defects. Apart from a deficiency in the use of English many theorems are ill-explained and even wrong unless interpreted exactly. The blame for this situation must be attributed to the editors of the journal. In this paper our aim is to state all his results carefully and explain the proofs in detail. Moreover using his methods we shall generalise his main theorem.

We consider the free product $G * \langle t \rangle$ of G with an infinite cyclic group generated by t. Let $w \in G * \langle t \rangle$. Then w has a unique expression of the form $g_0 t^{q_1} g_1 t^{q_2} \dots g_{n-1} t^{q_n} g_n$ where $g_i \in G$ are non-trivial for $0 < i < n$ and q_i are non-zero integers for each i. We call $\sum_{i=1}^{n} q_i$ the *exponent sum* of t in w, denoted $\mathrm{ex}(w)$, and $\sum_{i=1}^{n} |q_i|$ the *t-length* of w. The unreduced word $t^{q_1} t^{q_2} \dots t^{q_n}$ obtained by deleting the elements of G from w is called the *t-shape* of w.

Let $\langle\langle w \rangle\rangle$ denote the normal closure of w in $G * \langle t \rangle$.

The two main problems with which we shall be concerned are the Kervaire problem and the adjunction problem, which we now state.

THE KERVAIRE PROBLEM. *Suppose G is a non-trivial group. Is it possible for $\frac{G * \langle t \rangle}{\langle\langle w \rangle\rangle}$ to be trivial?*

The negative answer to the Kervaire problem is known as the *Kervaire conjecture* after a conversation between M. Kervaire and G. Baumslag c. 1963 [Ke, p. 117, MKS p. 403] and it has been proved for large classes of groups, for example compact topological groups, locally residually finite groups [GR, Ro], locally indicable groups [H_2, Sh]. In general however the problem is still open.

Now think of $w = 1$ as an equation in the "variable" t with "coefficients" the elements g_i. We say that $w = 1$ *has a solution over G* if there exists a group $\tilde{G}$ with G embedded in $\tilde{G}$ and an element $x \in \tilde{G}$ such that $w(x) = 1$ where $w(x)$ is the result in $\tilde{G}$ of substituting x for t in w. It is easily seen that $w(x) = 1$ has a solution over G if and only if the natural map $G \to \frac{G * \langle t \rangle}{\langle\langle w \rangle\rangle}$ is injective.

THE ADJUNCTION PROBLEM. *Under what circumstances does the equation $w = 1$ have a solution over G?*

Clearly it is necessary that $w \in G * \langle t \rangle - G$. However even if $w \notin G$ then G may still fail to embed in $\frac{G * \langle t \rangle}{\langle\langle w \rangle\rangle}$ as the following example shows. Let G be the cartesian product of a cyclic group of order p, generated by x, with a cyclic group of order q, generated by y, where p and q are coprime integers and let $w = xt^{-1}yt$ then $\frac{G * \langle t \rangle}{\langle\langle w \rangle\rangle}$ is infinite cyclic and G fails to embed.

Notice that this example has exponent sum zero and that the group G has torsion. If $\mathrm{ex}(w) \neq 0$ then the adjunction problem is also open in general. It is known that a solution exists in the case $\mathrm{ex}(w) = 1$ for the same classes of groups as for the Kervaire conjecture (listed above). Indeed a positive answer to the adjunction problem when $\mathrm{ex}(w) = 1$ would imply the Kervaire conjecture, since if $\mathrm{ex}(w) \neq \pm 1$ then $\frac{G * \langle t \rangle}{\langle\langle w \rangle\rangle}$ has a non-trivial cyclic quotient. It is also known that a solution exists if $|\mathrm{ex}(w)|$ is the t-length of w [L] and if $\mathrm{ex}(w) \neq 0$ and t-length $\leqslant 4$ [H_1, EH]. The problem considered here is a special case of the more general adjunction problem considered by Neumann, [N], which considers the effect of adding finitely many new generators and relators.

The main result proved here solves both problems when G is torsion-free for a large class of words, the *amenable* words. These are words whose t-shape satisfies a technical condition, and includes all words with $\mathrm{ex} = \pm 1$, for details see section 5.

THEOREM 1.1. *Let G be a torsion-free group and let $w \in G * \langle t \rangle - G$ be an amenable word. Then $w = 1$ has a solution over G.*

COROLLARY 1.2. (Klyachko: The Kervaire conjecture for torsion-free groups.) *Let G be a non-trivial torsion-free group and let w be an element of $G * \langle t \rangle - G$. Then $\frac{G * \langle t \rangle}{\langle\langle w \rangle\rangle}$ is non-trivial.*

Klyachko's paper [Kl] contains the proof of theorem 1.1 in the case in which the exponent sum of t in w is 1. (As observed above, this is the case which implies the Kervaire conjecture.) In this paper we shall take a direct path to theorem 1.1, the proof of which is given in sections 4 and 5. Klyachko proves some further results on solving equations over groups in a variety of other circumstances, and for completeness we shall give these results in section 6.

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2. The crash theorems

In this section we prove the crash theorems, which are the main techniques used by Klyachko for his applications.

Some preliminary definitions. Let $p : \mathbf{R} \to S^1$ be the universal covering map of the circle given by $p(t) := e^{2\pi i t}$, $t \in \mathbf{R}$. A function $f : \mathbf{R} \to S^1$ is called *proper, monotone* or *strictly monotone* if its lift $\tilde{f} : \mathbf{R} \to \mathbf{R}$ is proper, monotone or strictly monotone. A monotone function is called *anticlockwise* if its lift is increasing and *clockwise* if its lift is decreasing.

Now let K be a cell complex subdividing the 2-sphere. We shall assume that the 2-sphere has the usual orientation and that each 2-cell is oriented so that its attaching map is anticlockwise.

To help explain the crash theorems we will call each 2-cell a *country*, each 1-cell a *road* and each 0-cell a *junction*. Let $\phi_c : S^1 \to \partial c$ denote the attaching map of a country c. A *traffic flow* on K is defined to be a set of proper, monotone, anticlockwise functions $\{f_c : \mathbf{R} \to S^1\}$, one for each country c in K. We will think of $t \in \mathbf{R}$ as a time variable and the point $\kappa_c(t) := \phi_c \circ f_c(t)$ as the position of a car, belonging to c, on the boundary of c at time t. We will say that a car is *on* the road r if it is in the interior of the 1-cell r. The *order* of a junction is the number of ends of roads which are at that junction.

If two or more cars (from neighbouring countries) occupy the same point on a road or the same junction at the same time t, then a *crash* is said to occur at time t.

A *complete crash* occurs if either:

(1) Two cars (from neighbouring countries) occupy the same point on a road at the same time. This is called a *road crash*.

(2) n cars (from all the neighbouring countries) occupy a junction of order n at the same time. Note that it is possible for $n = 1$ so that, paradoxically, a complete crash may involve only one car (crashing into the end of a dead-end road)!

We would like to talk about traffic flows being in "general position". Such a flow would mean that no two cars are at a junction at the same time. There is an obvious notion of a "nearby" flow in which the motion is changed by an amount uniformly less than some positive but small number. However it is important that the result does not increase the number of crashes. The precise statement of the result we need is the following:

LEMMA 2.1. *Suppose for a traffic flow there is an interval of time* $t_0 \leqslant t \leqslant t_1$ *with no complete crashes in some open region* R *of the sphere. Suppose further that cars in* R *are at junctions for just one moment* s, *where* $t_0 < s < t_1$. *Then there is a nearby traffic flow, which is unaltered outside* R *and outside the time interval* $t_0 < t < t_1$, *with no crashes in* R *for* $t_0 \leqslant t \leqslant t_1$ *and such that no two cars in* R *are at junctions at the same time* t *for* $t_0 \leqslant t \leqslant t_1$.

Proof. Suppose there are a number of junctions involved. Then by a small change we can assume that the cars which arrive at the different junctions arrive at different times, without introducing any new crashes, complete or otherwise. So now restrict attention to one junction J in R and consider a small neighbourhood N of J in R. Since by hypothesis there is no complete crash at J for $t_0 \leqslant t \leqslant t_1$ we can assume that the number of cars which meet at J at time s is less than the order of J. A car arriving at a junction turns left. Choose a car κ so that the left turn at J leads to a road whose intersection with N has no car on it. Now hurry κ along so that it arrives ahead of the other cars and completes the turn first. Repeat this process for the remaining cars so that no crashes of any kind occur in N (and note that no new crashes have been introduced outside of N). By adjusting the speeds afterwards we can assume that the flow is unaltered outside of the time interval given. □

THEOREM 2.2. (Basic Crash Theorem.) *Let* K *be a cell decomposition of the 2-sphere with at least one 1-cell. Then any strictly monotone traffic flow on* K *has at least 2 complete crashes at two different places.*

Proof. The hypothesis that K contains at least one 1-cell implies that each 2-cell is attached to one or more 1-cells and that at nearly all times cars will be disjoint and away from junctions. Let t_0 be such a time. Construct an oriented graph Γ, embedded in the 2-sphere, called the *cross-traffic graph,* by the following procedure. (It may be helpful for the reader to consult figure 1 at this point. The cross traffic graph is in heavy print.) For each country c pick some point in the interior as its *capital* C. The capitals will form the vertices of Γ. Suppose that c's car κ_c is on the road forming a common boundary with country c'. Join the capitals C and C' by an edge of Γ oriented from C to C' passing radially outwards in c from C to κ_c and then radially inwards in c' to C'. (It may happen of course that c is c' and so the edge is a loop.)

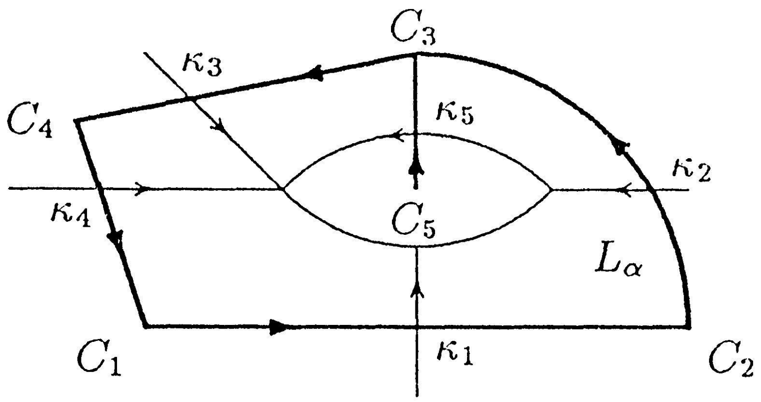

FIGURE 1
The construction of the cross-traffic graph

Notice that Γ is essentially a subgraph of the dual cell subdivision.

Since every capital has an exit path in Γ it follows that Γ will contain coherently oriented simple closed curves. If α is such a simple closed curve let L_α denote the disc with boundary α which is on the left of α when traversed in the direction of orientation. Similarly denote the complementary disc by R_α. Traffic flows *into* L_α as time progresses and *out* of R_α.

Let D be a minimal nested disc amongst the discs L_α and R_α for all loops α of the cross-traffic graph. We shall prove that a complete crash occurs at some time in the interior of D. Since there are at least two such minimal discs (with disjoint interiors) this proves the theorem.

For definiteness assume that $D = L_\alpha$ for some α and watch what happens as time flows forwards. (If $D = R_\alpha$ then we let time flow backwards.) As time proceeds either a road crash occurs or D shrinks upon itself in a continuous fashion until some car inside D comes to a junction at time s say. At this point we have to redefine Γ.

Either there is a complete crash inside D at time s (as required) or by the lemma we can assume that the cars in D arrive at junctions one at a time and we consider the new graph Γ after the first car has passed a junction. There are two possibilities, either the car involved is part of the circuit α or it is not.

If the car is part of α then the corresponding edge breaks the circuit and passes inside D and eventually gives a new circuit defining a new innermost disc inside D. (It can be checked that this is again an L_β for some β.)

If the car is not part of α then either D is still minimal and we proceed or we now have a new minimal disc inside D and we again proceed (in fact the minimality of D implies that the edges of Γ inside D form a forest and it can then be checked that this latter case is impossible, but we shall not need to do this).

Eventually we arrive at a situation where α comprises just one or two edges. In the first case α is a loop around a dead-end junction and a complete crash occurs there and in the second case two cars are approaching each other either on the same road or on two roads with a common junction of order 2 and a complete crash occurs. $\square$

REMARK. In fact it can be seen that there must be infinitely many complete crashes and moreover we can find a subset of these crashes occurring at times t_i, $i \in \mathbf{Z}$ with $t_i < t_{i+1}$ such that, for each i, the crash at time t_i is at a different place to the crash at time t_{i+1}.

TRAFFIC FLOWS WITH STOPS

If we consider traffic flows on a cell decomposition of the 2-sphere which are monotone but not *strictly* monotone, i.e. have *stops*, then it is possible to avoid complete crashes. The following example should make this clear. Consider a neighbourhood of a junction O which we take as the origin and four roads joining O which we take as the coordinate axes. As usual let the increasing direction of the x-coordinate be from west to east and the increasing direction of the y-coordinate be from south to north. Suppose now that there are four cars E, N, S, W approaching O along these roads which in the normal course of events would have a complete crash at O. If stops are allowed then complete crashes can be avoided as follows.

Let E, N and S come to O and crash (incompletely) while W slows down. Now whilst N and S stop at O let E continue south. Now W comes to O and another incomplete crash occurs. The cars can now continue their journey and by adjusting their speeds accordingly can be made to travel as though nothing had happened.

The problem here was that the two stopped cars N and S are next to one another if you ignore E and W.

We shall need to assume that cars which stop at a given vertex do so *each* time they visit that vertex. The following definition for such a traffic flow with stops avoids the problem described above and allows a generalisation of theorem 2.1 to be proved.

DEFINITION. Let v be a junction and let $c_1, \ldots, c_n$ be the countries, listed in anti-clockwise order about v, whose cars $\kappa_1, \ldots, \kappa_n$ actually stop for a positive time at v. Let T_i be the union of the intervals of time that κ_i stops at v. We say that the flow has *separated stops* at v if, for the stopping countries c_i, c_j where $|i - j| = 1 \bmod n$, the unions of intervals T_i and T_j

are disjoint. (Note that under these circumstances, more than one car stops at v and there cannot be a complete crash at v.)

We can now prove a generalisation of the original crash theorem in which cars are allowed to stop.

THEOREM 2.3. (Crash Theorem with Stops.) *Let K be a cell decomposition of the 2-sphere with at least one 1-cell. Then any monotone traffic flow on K with separated stops at each stopping vertex has at least 2 complete crashes at two different places.*

Proof. Let us use the notation developed above. So v is a junction and $c_1, \ldots, c_n$ are the countries, in anticlockwise order about v, whose cars $\kappa_1, \ldots, \kappa_n$ actually stop for a positive time at v. The idea is to change K by blowing up each such junction v to a disc D and defining a strictly monotone traffic flow on a new subdivision K'. This is done as follows. Define the portion of K lying in the interior of D to be a new country. The boundary of D is naturally subdivided into junctions (of order 3) and roads by intersection with the countries adjacent to v. Now collapse to junctions all roads of the boundary of D which are on the boundary of a country whose cars do *not stop* at v (see figure 2).

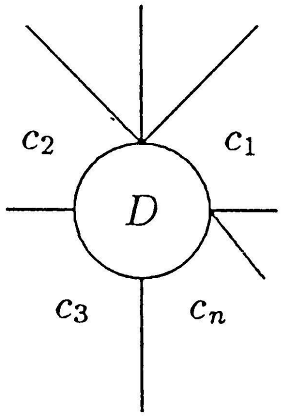

FIGURE 2
The construction of the new cell complex K'

The motion of the original cars which stopped at v can be extended to K' without stops by having them move monotonically along the boundary of D during the time when they originally would have been stopped. The motion of the original cars which do not stop at v is extended to K' in the obvious way.

Now we define the motion of a car κ_D in an anticlockwise manner around the boundary of D. This will be done in such a manner that no complete crashes occur on the boundary of D. We will use the following notation. Let r_i be the road common to the boundary of D and the country with stopping car κ_i. Let the end junctions of r_i be v_i and v_{i+1} in anticlockwise order $i = 1, 2, \ldots$. Suppose κ_i is on the road r_i. Then by hypothesis the roads r_{i+1} and r_{i-1} are free of the cars κ_{i+1} and κ_{i-1} respectively (see figure 3).

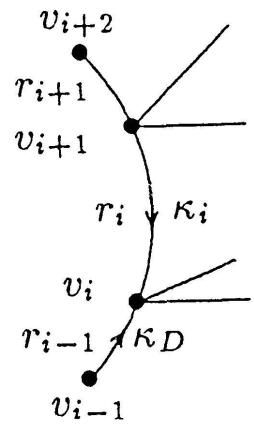

FIGURE 3
The motion of car κ_D

As κ_i traverses r_i from v_{i+1} to v_i let κ_D traverse r_{i-1} from v_{i-1} to v_i. Let the cars meet at v_i at time t. This will not be a complete crash since κ_{i-1} is missing. Again by hypothesis κ_{i+1} will not be at v_{i+2} at time t. Let r be largest such that κ_{i+r} is not at v_{i+r+1} at time t. Then κ_D has enough time to arrive at v_{i+r+1} just as κ_{i+r} does. If there is no such r then let κ_{i+r} be the next car to arrive at D and let κ_D go once round the entire boundary and arrive at v_{i+r+1} just as κ_{i+r} does. Keep repeating this strategem to define the motion of κ_D.

Now we are in a situation corresponding to the first crash theorem and the result is proved. $\square$

3. TWO TRANSVERSALITY LEMMAS

In this section we use transversality (cf. [BRS, F]) to prove the existence of diagrams of van-Kampen type for the two situations that we shall meet in the applications to group theory of the crash theorems (in sections 4, 5 and 6). These lemmas need to be stated very carefully and a failure to do so is one of the major weaknesses in Klyachko's version. The lemmas use the idea of a *corner* of a 2-cell in a cell subdivision K of the 2-sphere. This can be regarded as the (oriented) angle formed by the two adjacent edges meeting at a 0-cell in the boundary of the 2-cell. If all the corners of a 2-cell are labelled by elements of a group, then a word can be read around the 2-cell boundary by composing these elements either unchanged or inverted according as the orientation of the corner agrees or disagrees with that of the 2-cell boundary. Similarly if all the corners at a 0-cell are labelled then a word can be read around that 0-cell. We shall always orient corners *clockwise,* thus if the above words are read *clockwise* for 0-cells and *anticlockwise* for 2-cells, then no inversion is necessary (see figure 4).

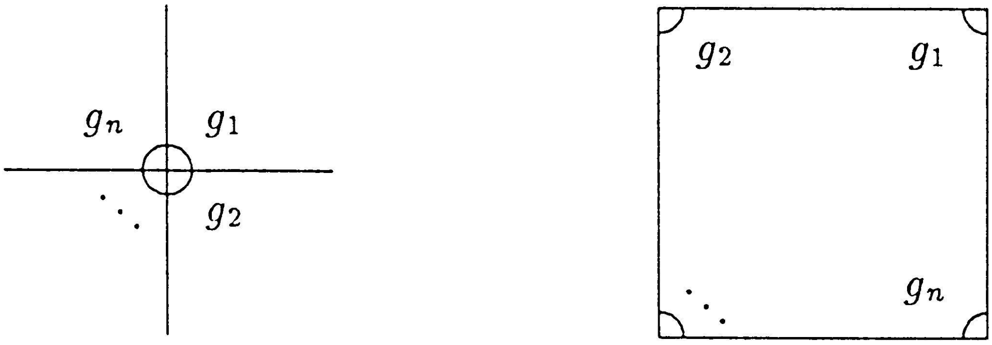

FIGURE 4
Multiplying the corner labels to get $g_1 g_2 \cdots g_n$ for a 0-cell and a 2-cell

Let $w \in A * B$ be an element in the free product of two groups. We shall only be interested in w up to cyclic reordering and thus (cyclically reordering if necessary) we can assume that either $w = 1$ or $w \in A \cup B - 1$ or w is written uniquely as $a_1 b_1 a_2 b_2 \cdots a_n b_n$ where a_i and b_i are non-trivial elements of A, B alternately. These non-trivial elements a_i, b_i are then called the (cyclic) *factors* of w.

LEMMA 3.1. *Let A, B be two groups and let $N = \langle\langle W \rangle\rangle$ be the the normal closure in $A * B$ of some subset of elements $W \subset A * B$. Suppose $N \cap A \neq \{1\}$. Then there is a cell subdivision of the sphere S^2 such that each corner of each 2-cell is labelled by an element of $A \cup B$ with the following properties.*

1. The corner labels of a 2-cell are the cyclic factors (in anticlockwise order and up to cyclic rotation) of some w or w^{-1} where $w \in W$.

2. The corner labels at a 0-cell are either all in A or all in B.

3. The (clockwise) product of the corner labels at a 0-cell is 1 (in A or B) except for one special 0-cell where the product is a non-trivial element of $A \cup B$.

Proof. Let K_A, K_B be two disjoint 2-dimensional complexes such that $\pi_1(K_A, *_A) = A$ and $\pi_1(K_B, *_B) = B$. Join the base points $*_A$ and $*_B$ by an arc α with central point $*$. Let $K = K_A \cup \alpha \cup K_B$. Then $\pi_1(K, *)$

$\cong A * B$. Attach 2-cells σ_w to K by the words $w \in W$ to form the complex L. If $a \in N \cap A - \{1\}$ there is a map $f: D^2, S^1 \to L, K$ from the 2-disc to L such that the restriction $f \mid S^1$ to the boundary represents a. Make the map f transverse to the centres of the 2-cells σ_w. It follows that the inverse images of small neighbourhoods of these centres is a collection of disjoint discs $D_1, \ldots, D_m$ in the interior of D^2. By a radial expansion of f on these discs we may assume that each image is the whole of one of the σ_w. It follows that the punctured disc $P = D^2 - \overline{D_1 \cup \cdots \cup D_m}$ is mapped by f to K. Make $f \mid P$ transverse to $*$. Then $f^{-1}*$ is a 1-manifold Z properly embedded in P. By a radial expansion along α we can assume that Z has a neighbourhood N which is a normal I-bundle and where each fibre is mapped by f to α. The complementary space $P - N$ is divided into connected regions which are mapped by f to K_A or K_B. On crossing N one passes from one kind of region to the other.

We now simplify the subset $D_1 \cup \cdots \cup D_m \cup N$ of D^2 as follows. Suppose N contains an annulus component $\mathscr{A}$ in the interior of P. Let D' denote the interior disc of D^2 which bounds the interior boundary component of the annulus. Then $D' \cup \mathscr{A}$ is a sub disc of D^2 whose boundary gets mapped to a base point by f. We can then shrink it to a point, redefine f and simplify the situation. Having eliminated all annuli, $D_1 \cup \cdots \cup D_m \cup N$ will look like a thickened graph in D^2 with the discs D_i corresponding to thickened vertices and the components of N to thickened edges. Our next task is to make this graph connected. If not choose an innermost component C. Draw a simple loop around C separating it from the rest of $D_1 \cup \cdots \cup D_m \cup N$. This loop will represent (up to conjugacy) an element of $A \cup B$. If this element is trivial we can shrink the disc it bounds as above and simplify the situation. If not we replace $D_1 \cup \cdots \cup D_m \cup N$ by C. Note that the boundary curve may now represent a non trivial element of B instead of A.

Attach a 2-cell (outside) to the boundary of D^2 and label the centre of this outside cell ∞. The 2-disc has now become a 2-sphere. In this situation consider the dual graph Γ. This has a vertex in each region and an edge joining neighbouring regions separated by a component of N. For the outer region take the vertex to be ∞. Then Γ and its complementary regions define a cell subdivision K of the 2-sphere. Each vertex is either in an A region or a B region and the corners can be correspondingly labelled by elements of A or B as follows. Every 2-cell of K contains a unique subdisc D_i. Opposite a corner is an edge of D_i labelled by an element of A or B. Take this to be the labelling of the corner. By moving anticlockwise around the boundary of a 2-cell of K the corner labellings spell out a cyclic rotation of some w_i or w_i^{-1}. By moving clockwise around a 0-cell of K the corner labellings spell out the trivial element (of A or B) except for ∞ which spells out a non-trivial element of A or B. $\square$

NOTE. It may not be possible to specify that the non-trivial element lies in A as this simple example shows. Let $A = \langle a \rangle$, $B = \langle b \rangle$ be two infinite cyclic groups generated by a, b respectively. Let the words of the attaching 2-cells be ab^{-1}, b. In this case the 2-cells of the required subdivision have either two corners (those modelled on ab^{-1}) or one corner (modelled on b) and the only possible subdivision of the 2-sphere satisfying lemma 3.1 is the trivial one with single vertex labelled b. This is a place where Klyachko's version is definitely wrong (rather than badly stated).

Let $w \in G * \langle t \rangle$ be an element of the free product of a group G with the infinite cyclic group $\langle t \rangle$. Then w can be written uniquely (up to cyclic rotation) in the form $w = g_1 t^{\varepsilon_1} g_2 \cdots t^{\varepsilon_n}$ where each $g_i \in G$, each $\varepsilon_i = \pm 1$ and g_i can only be 1 if it has neighbouring t's (in cyclic order) with the same exponent. We call $g_1, \ldots, g_n$ the *coefficients* of w.

The following lemma is proved in [H$_1$]. It is closely related to "pictures" [R$_1$, R$_2$, Sh].

LEMMA 3.2. *Let G be a group and consider the free product $G * \langle t \rangle$ of G with an infinite cyclic group (generator t). Let $N = \langle\langle W \rangle\rangle$ be the the normal closure in $G * \langle t \rangle$ of some subset of elements $W \subset G * \langle t \rangle$. Suppose $N \cap G \neq \{1\}$ then there is a cell subdivision K of the 2-sphere such that*

a) *the 1-cells of K are oriented,*

b) *the corners (all oriented clockwise) are labelled by coefficients of elements of W,*

c) *the clockwise product of the corner labelling around any 0-cell is 1 except for one vertex where it is non trivial,*

d) *the corner labels of any 2-cell (in anticlockwise order) are the coefficients of w or w^{-1} for some $w \in W$ (up to cyclic rotation) with the property that, if on passing from one corner to an adjacent corner the element t or t^{-1} is inserted according to whether the intervening edge is oriented in the same or opposite direction, then the whole of w or w^{-1} is recovered.*

Proof. The proof is very similar to 3.1. Let K_G be a 2-dimensional complex such that $\pi_1(K_G, *_G) = G$. Adjoin an oriented 1-cell γ to the base point $*_G$ to form a 2-dimensional complex $K = K_G \vee S^1$ with $\pi_1 K = G * \langle t \rangle$. Attach 2-cells to K by the words $w \in W$ to form L. Since $N \cap G \neq \{1\}$ there is a non contractable loop in K_G represented by a map $f: S^1, 1 \to K_G, *_G$ which can be extended to a map $f: D^2 \to L$.

We now proceed as in the proof of lemma 3.1 with the rôle of $*$ played by a point p in the interior of γ. We construct a graph whose (thickened) vertices are the inverse image of the 2-cells and whose edges are the inverse image of p. By making similar simplifications and passing to an innermost component, as before, we may assume that this graph is connected. Replace D^2 by a sphere as before. The dual subdivision now defines K. The orientation of the 1-cells is determined by the orientation of γ and it only remains to observe that these oriented edges correspond to the new generator t. $\square$

4. Application to the Kervaire problem

In this section we give Klyachko's application of the crash theorems to prove theorem 1.1 in the case in which exponent sum of t in the word w is 1. As remarked in the introduction this implies the Kervaire conjecture for torsion-free groups.

We say that a system of equations $\{w(t) = 1 \mid w \in W\}$ in the variable t, with coefficients in a group G, has a *solution over* G if there is a group $\tilde{G}$ containing G as a subgroup and an element $x \in \tilde{G}$ such that the relations $\{w(x) = 1 \mid w \in W\}$ are satisfied in $\tilde{G}$. It is clear that this is equivalent to the natural map

$$G \to \frac{G * \langle t \rangle}{\langle\langle W \rangle\rangle}$$

being injective, where $\langle\langle W \rangle\rangle$ denotes the normal closure of W in $G * \langle t \rangle$.

Now let H be a subgroup of G and let $g \in G$. We say that g is *free relative* to H if the subgroup $\langle g, H \rangle$ of G generated by g and H is naturally the free product $\langle g \rangle * H$ of an infinite cyclic group $\langle g \rangle$ with H.

We shall apply the crash theorem with stops to prove theorem 4.1 (below) and then use an algebraic trick to deduce the case $\mathrm{ex}(w) = 1$ of theorem 1.1.

If g, h are elements of a group let g^h denote $h^{-1}gh$.

THEOREM 4.1. *Let* H *and* H' *be two isomorphic subgroups of a group* Γ *under the isomorphism* $h \to h^{\phi}, h \in H$. *Suppose that for each* i, a_i, b_i *are elements of* Γ *such that* a_i *is free relative to* H *and* b_i *is free relative to* H'. *Let* c *be an arbitrary element of* Γ. *Then the system of equations*

$$(b_0 a_0^t b_1 a_1^t b_2 a_2^t \cdots b_r a_r^t) c t = 1 \tag{1}$$

$$h^{\phi} = h^t, h \in H \tag{2}$$

has a solution over Γ.

Proof. Assume not. Then by the second transversality lemma there is a cell subdivision of the 2-sphere such that, the 1-cells of K are oriented, the 2-cells are of the four types I, I', II and II' illustrated in figure 5 with the corners labelled by elements of G as shown and such that the clockwise product of the corner labelling around any 0-cell is 1 except for one vertex (where it is non-trivial). Assume that K is minimal with these properties.

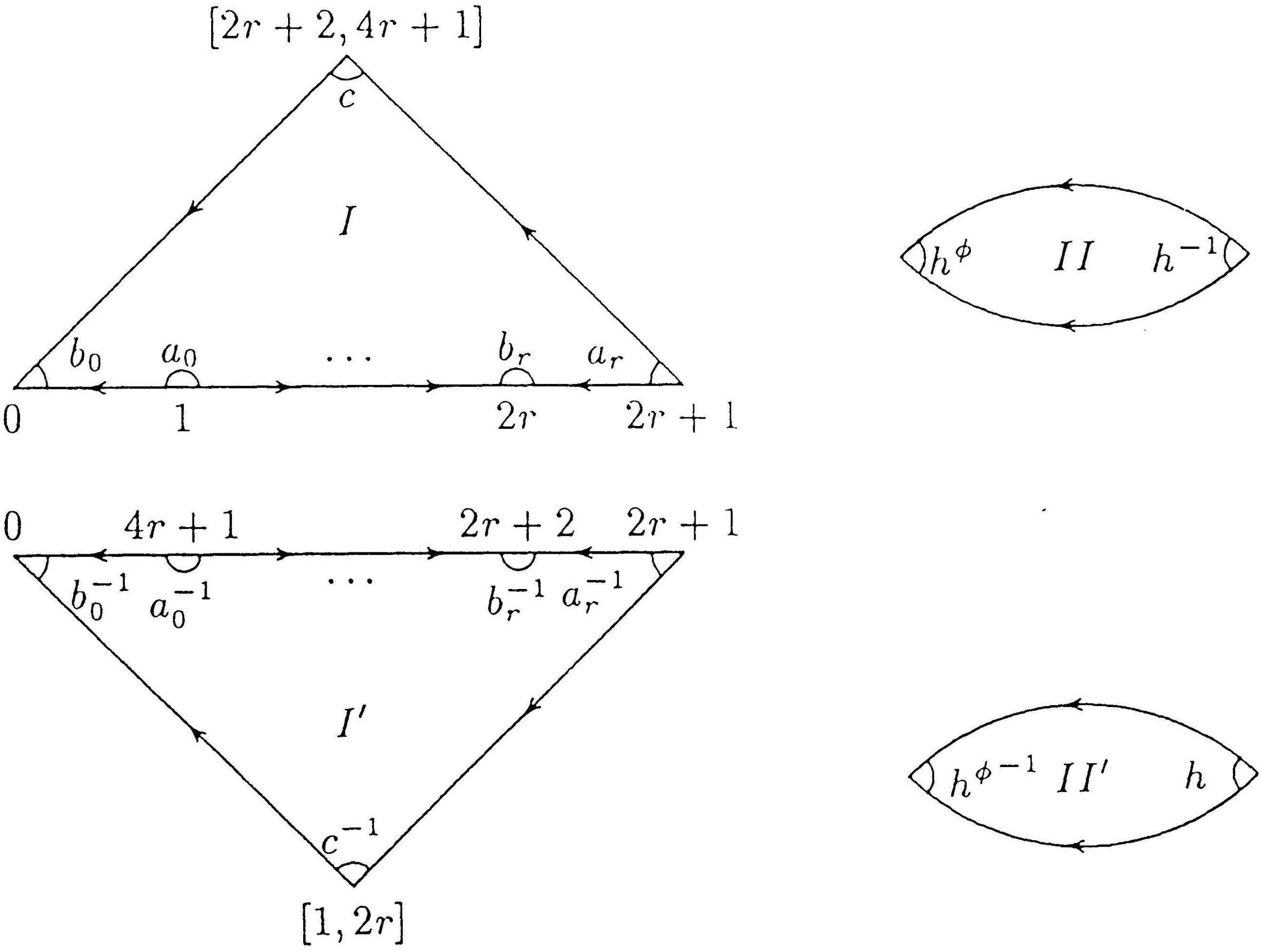

FIGURE 5
The 2-cells I, I', II and II'

A traffic flow is now defined on K as follows. At time 0 let a car on the boundary of a country of type I or I' start at the corner labelled b_0 or b_0^{-1} and proceed in an anticlockwise manner with respect to the orientation of the edge along which it is travelling, moving from corner to corner in unit time except at the corner labelled c or c^{-1} where it stops for $2r - 1$ units. The times when the car is at each corner are illustrated in figure 5. For countries of type II or II' the car starts at the corner labelled h^{Φ} or $h^{\Phi^{-1}}$ and proceeds in an anticlockwise manner moving from corner to corner in unit time.

The fact that the edges are oriented will be used to derive various properties of this flow and of the cell subdivision K. We shall think of the orientation arrows as pointing uphill so that corners come in four types: *top corners,* of which the corner labelled b_0 in 2-cells of type I (see the figure) is an example; *bottom corners*, for example the corner labelled a_0; *up corners*, for example c, and *down corners*, for example c^{-1}. Similarly the 0-cells of K come in three types: *maxima* or sinks, where all the corners are top corners, *minima* or sources, where all the corners are bottom corners and *saddles*, where some of the corners are uphill or downhill. Notice that at a saddle the up and down corners are equal in number and must alternate around the 0-cell (figure 6) although they may be separated by top or bottom corners.

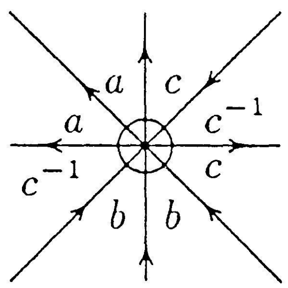

FIGURE 6
Up and down corners alternate around a 0-cell

PROPERTY 1: ONE-WAY FLOW. *If two cars are on roads at the same time then they are both either moving uphill or both downhill.*

Indeed, in intervals of time of the form $(2n, 2n + 1)$, no car is moving uphill and and in intervals of the form $(2n - 1, 2n)$ no car is moving downhill.

Property 1 implies that there are no road crashes.

PROPERTY 2: STOPPING SCHEDULE. *Cars always stop at up and down corners but never at top or bottom corners.*

Property 2 implies that saddles are stopping vertices whilst maxima and minima are not.

PROPERTY 3: SEPARATED STOPS. *It never happens that there is one car at an up corner and at the same time another car at a down corner.*

Property 3 (together with the observation that up and down corners alternate around a saddle) implies that stops are separated, as required for the crash theorem with stops. It also implies that crashes can only occur at maxima or minima.

PROPERTY 4: COHERENCE. *If two cars in cells of type I or I' are both at a non-stop corner, then the corners carry the same label (possibly inverted).*

Thus if a crash occurs at a vertex with labelling giving a trivial product which is a minimum (resp. maximum) then this implies a relationship in $\langle a_i, H\rangle$, (resp. $\langle b_i, H'\rangle$) or that some power of a_i, b_i is trivial for some i. By the hypotheses of the theorem, this can only happen if there is a pair of adjacent corners labelled by a_i, a_i^{-1} or $h, h^{-1}, \ldots$. So there is a neighbouring pair of 2-cells of type I, I' or II, II' which can be removed, simplifying K. It follows that there can only be a total crash at the vertex with non-trivial labelling contradicting the crash theorem with stops which states that there are at least two. $\square$

REMARK. By taking H and H' to be trivial in theorem 4.1, we can now deduce a special case of theorem 1.1, namely the case when the t-shape of w is $t^{-1}tt^{-1}\ldots tt^{-1}tt$. The rest of the section introduces an algebraic trick which will enable us to reduce the general case $\mathrm{ex}(w) = 1$ to this special case.

Let G be a group and consider the homomorphism $\mathrm{ex}\colon G * \langle t\rangle \to \mathbf{Z}$. It is clear that K, the kernel of ex, is generated by elements of the form $g^{t^{\mathscr{O}}} = t^{-\mathscr{O}} g t^{\mathscr{O}}, g \in G$.

Any element of K has a canonical expression of the form $k = g_1^{t^{\mathscr{O}_1}} \cdots g_r^{t^{\mathscr{O}_r}}$, where $\mathscr{O}_i \neq \mathscr{O}_{i+1}$ for each i. We shall call the $g_i^{t^{\mathscr{O}_i}}$ the *canonical elements* of k. Let $\min(k)$ be the minimum value of $\mathscr{O}_i, i = 1, \ldots, r$ and $\max(k)$ the maximum value. Fix a positive integer m. Consider the following subgroups of K:

$$H = \langle k \in K \mid \min(k) \geqslant 0, \max(k) \leqslant m - 2\rangle$$
$$H' = \langle k \in K \mid \min(k) \geqslant 1, \max(k) \leqslant m - 1\rangle$$
$$J = \langle k \in K \mid \min(k) \geqslant 0, \max(k) \leqslant m - 1\rangle$$

and the following subsets:

$$X = \{k \in K \mid \min(k) = 0, \max(k) \leqslant m - 1\}$$
$$Y = \{k \in K \mid \min(k) \geqslant 0, \max(k) = m - 1\}$$
$$Z = \{k \in K \mid \min(k) \geqslant 1, \max(k) = m\} .$$

LEMMA 4.2. *Let* $w \in G * \langle t\rangle$ *satisfy* $\mathrm{ex}(w) = 1$. *Then, after conjugation,* w *can be written as a product*

$$b_0 a_0^t b_1 a_1^t \cdots b_r a_r^t c t ,$$

where $a_i \in Y$, $b_i \in X$, $i = 0, \dots, r$ *and* $c \in J$ *for some* m.

Proof. Clearly w can be written as a product kt where $k \in K$. If $k = 1$ the result is trivial. Otherwise let $k = \prod_1^N g_i^{t^{\mathscr{O}_i}}$ be the canonical expression for k. Conjugating by a suitable power of t, we can assume that $\min(k) = 0$. Let $m = \max(k)$. Since successive $\mathscr{O}_i$ differ, we must have $m \geqslant 1$. Now consider the appearances of the maximum m and the minimum 0 in the canonical expression. Suppose that the first appearance is a minimum. Then by collecting terms up to (but not including) the first maximum, we define an element of X which forms the left part of the canonical expression for k. If the first appearance is a maximum, then we would find an element of Z instead. Continuing in this way we can write k as a product of elements taken alternately from X and Z.

If the first element is from Z, i.e. $k = zxu$ where $z \in Z$, $x \in X$ and u is the rest of the canonical expression, then we conjugate w by z to yield $k't$ where $k' = xuz^{t^{-1}}$. Now the canonical expression for k' may be simpler than that for k and the max may have dropped by 1, but the min is still 0 and now the expression of k' as a product of elements taken alternately from X and Z starts with an element of X.

Thus we may assume that k can be written as a product

$$x_0 z_0 x_1 z_1 \dots x_r z_r c$$

where $x_i \in X$, $z_i \in Z$ and $c = 1$ or $c \in X$ (and notice that $c \in J$ in either case). Now let $b_i = x_i$ and $a_i^t = z_i$ then $w = kt$ has the required expression. $\square$

LEMMA 4.3. *Suppose that* G *is torsion free then any element* a *of* Y *is free relative to* H. *Similary any element* b *of* X *is free relative to* H'.

Proof. Suppose that a lies in Y. The case $b \in X$ is similar. Let $a = h_1 x_1 h_2 x_2 \cdots h_r$ where $x_i = t^{-m+1} g_i t^{m-1}$ and $h_i \in H$. We can assume that $g_1, \ldots, g_{r-1}$ and $h_2, \ldots, h_{r-1}$ are never the identity element. Assume there is a non trivial relationship $w(a, H) = 1$ which is minimal with respect to the number of occurences of a and its inverse a^{-1}. In particular no cancelling pairs aa^{-1} or $a^{-1}a$ can occur in w. Since G is torsion free the reduction of w to 1 can only occur as a sequence of elimination of pairs hh^{-1} where $h \in H$ or xx^{-1}, where x is an x_i or x_j^{-1}.

Now let $a' = x_1 h_2 x_2 \cdots x_{r-1}$ and define the *core* q of a by $a' = pqp^{-1}$ where p has maximal length. Then, in any subword of w of the form $ahah' \cdots$ where a occurs n times, at least every copy of q, together with the p in the first a and the p^{-1} in the last a, must survive after cancellation. Moreover in any subword of the form aha^{-1} since $h \neq 1$ cancellation can only involve combining h_r, h and h_r^{-1} after which no further cancellation is possible. The result is now clear. $\square$

THEOREM 4.4 (Case ex = 1 of theorem 1.1). *Let* G *be a torsion-free group and let* w *be an element of* $G * \langle t \rangle$ *with* $\mathrm{ex}(w) = 1$ *then the equation* $w = 1$ *has a solution over* G.

Proof. By lemma 4.2 we can assume that $w = b_0 a_0^t b_1 a_1^t \cdots b_r a_r^t c t$, where $a_i \in Y$, $b_i \in X$, $i = 0, \ldots, r$ and $c \in J$. We need to think of each a_i, b_i, c as functions of t and for clarity we shall introduce a new variable s. To be precise let

$$w(s, t) \equiv b_0(t) a_0^s(t) \cdots b_r(t) a_r^s(t) c(t) s$$

where s and t are independent variables.

Write Γ for $G * \langle t \rangle$ and let H, H' be the subgroups defined above. There is an isomorphism $\phi: H \to H'$ given by $h^\phi = h^t, h \in H$.

Lemma 4.3 gives the hypotheses of theorem 4.1, which implies that Γ embeds in $\tilde{\Gamma} = \langle \Gamma, s \mid w(s, t) = 1,\ h^s = h^\phi,\ h \in H \rangle$. Now each of the canonical elements of $a_i(t), b_i(t), c(t)$ is either in G or lies in H^t; moreover in $\tilde{\Gamma}$ we have $h^s = h^\phi = h^t$ for each $h \in H$. It follows that $w(s, s) = 1$ in $\tilde{\Gamma}$.

Therefore there is a commuting diagram

$$\begin{array}{ccc} \Gamma & \subset & \tilde{\Gamma} \\ \cup & & \uparrow \\ G & \to & \hat{G} \end{array}$$

Where $\hat{G} = \frac{G * \langle s \rangle}{\langle\langle w \rangle\rangle}$. Thus $G \to \hat{G}$ is injective. $\square$

REMARK. The alert reader will have noticed that the hypotheses of theorem 4.4 can be weakened. All that has been used is that the coefficients of w are of infinite order. Indeed a careful examination of the proof yields the following sharper statement. If $G \to \frac{G * \langle t \rangle}{\langle\langle w \rangle\rangle}$ is not injective then one of the *separating* coefficients in w has finite order (separating means between a t and a t^{-1}).

5. THE GENERAL CASE

In this section we consider the adjunction problem as stated in the introduction, in its full generality. We continue to work with torsion-free groups. We shall introduce a class of words with exponent not necessarily 1 for which the methods of the previous sections can be adapted to provide a solution to the adjunction problem. We call such words *amenable*. Before defining amenability in general we shall consider a class of simpler words, on which the general definition will be based, these we call *suitable* words.

t-SHAPES, t-SEQUENCES AND SUITABILITY

Consider finite sequences whose elements are t or t^{-1}. We call such a sequence a *t-sequence.* If m is a positive integer let t^m denote the sequence $t, t, \ldots, t$, m times and let t^{-m} denote the sequence $t^{-1}, t^{-1}, \ldots, t^{-1}$, m times. A *clump* is a maximal connected subsequence of the form t^m or t^{-m} where $m > 1$ and these are said to have *order m* and $-m$ respectively. We call a clump of positive order an *up* clump and a clump of negative order a *down* clump. A sequence is *suitable* if it has exactly one up clump which is not the whole sequence and possibly some down clumps, or if it has exactly one down clump which is not the whole sequence and possibly some up clumps.

It follows that, after a possible cyclic rotation or change $t \mapsto t^{-1}$, a suitable sequence has the form

$$t^s t^{-r_0} t t^{-r_1} t \ldots t t^{-r_k}$$

where $s > 1$, $k \geqslant 0$ and $r_i \geqslant 1$ for $i = 0, \ldots, k$. Notice that the power sequences t^n, $n > 1$ and the alternating sequences $tt^{-1}tt^{-1} \ldots tt^{-1}$ are not suitable.

Any word in $G * \langle t \rangle$ has a t-sequence associated to it, given by its t-shape. Since we are only interested in words up to cyclic permutation we shall say that a word is *suitable* if after a cyclic permutation its associated sequence is suitable.

THEOREM 5.1. *Suppose that* G *is torsion-free and that* $w \in G * \langle t \rangle$ *is a suitable word. Then* w *has a solution over* G.

Proof. We shall prove the theorem directly without using the algebraic trick used in the last section. Suppose that G does not inject in $\frac{G * \langle t \rangle}{\langle\langle w \rangle\rangle}$. By the second transversality lemma there is a cell subdivision of the 2-sphere such that, the 1-cells of K are oriented, the 2-cells are of the two types I and I' illustrated in figure 7 with the corners labelled by elements of G and such that the clockwise product of the corner labelling around any 0-cell is 1 except for one vertex (where it is non-trivial). Assume that K is minimal with these properties.

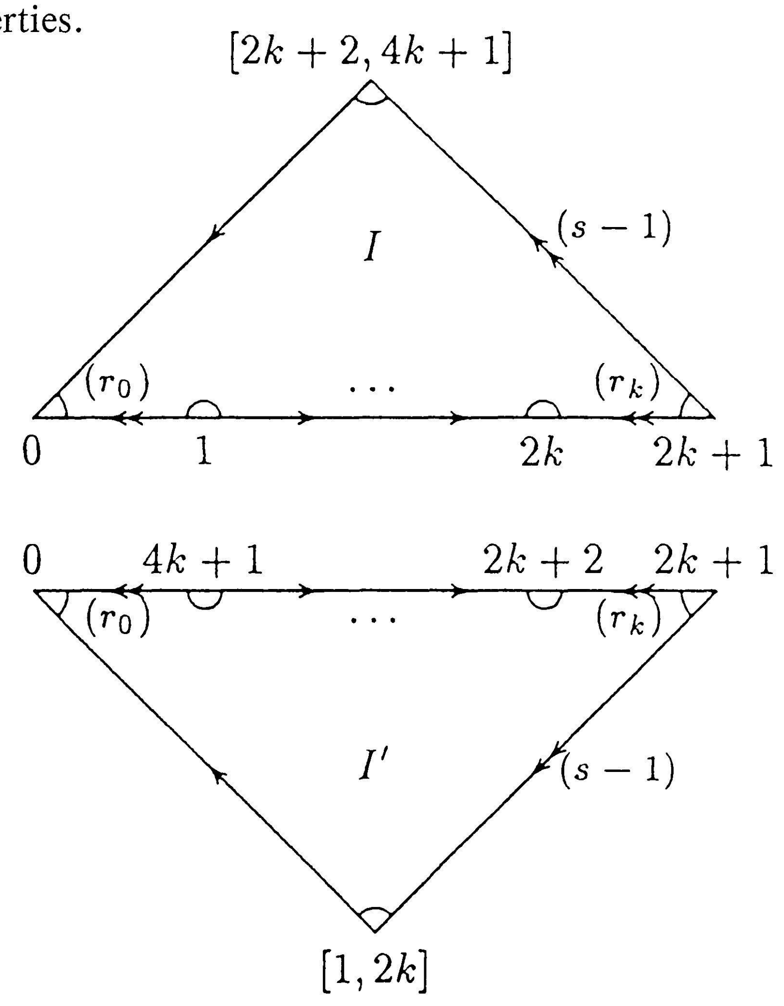

FIGURE 7
The 2-cells I, I'

In figure 7 we have used double arrows labelled by (r) as an abbreviation for r consecutive single arrows (corresponding to t^r). For example the top right double arrow in cell I represents $s-1$ single arrows.

We shall organise a traffic flow on K as in the proof of theorem 4.1 and having precisely the same four properties that were listed there. Then there can only be complete crashes at maxima or minima as before and such a crash at a vertex with trivial labelling implies a simplification of the diagram, just as before.

The traffic flow is defined in a very similar way to the flow in theorem 4.1. The unit times for the motion are given in figure 7. We have to explain what happens inside the clumps. The idea is to treat each of these as a uphill or downhill section along which a car travels in unit time pausing very briefly at each internal (stopping) vertex. The exception is the one up clump in cells of type I, where the car parks at the last possible stopping vertex while most of the motion takes place (in the other half of the cells of type I'), with a similar (reversed) motion in cells of type I'. □

We now combine the theorem with the algebraic trick of the last section.

NORMAL FORM AND AMENABILITY

There is a definition of a *normal form* for a word based on any t-sequence, which is similar to the form of lemma 4.2 (which can be regarded as defining the normal form for the t-sequence $t^{-1}tt^{-1}t\ldots t^{-1}t^2$).

To be precise, the normal form associated to a t-sequence is obtained as follows: arrange the t's and t^{-1}'s in the given sequence in anticlockwise order around a circle with a vertex between each pair. Put arrows anticlockwise next to each t and clockwise next to each t^{-1}. The vertices are now of the four types (top, bottom, up, down) defined in the proof of theorem 4.1. Write a letter c next to each up and each down vertex, a letter b next to each top vertex and a letter a next to each bottom vertex. Reading the letters round the circle anticlockwise (starting anywhere) gives the required normal form, where the letters a, b, c are interpreted as generic elements of Y, X, J respectively, where X, Y, J have the same meanings as in lemma 4.2.

A word is *amenable* if it can be conjugated to a word in normal form for a *suitable* t-sequence (as defined above the theorem). Notice that amenability is again a property of the t-shape of a given word and that lemma 4.2 proves that all words of exponent ± 1 are amenable.

THEOREM 5.2 (General case of theorem 1.1). *Suppose that* G *is torsion-free and that* $w \in G * \langle t \rangle$ *is an amenable word. Then* w *has a solution over* G.

Proof. The proof is a straightforward combination of the proof of theorem 5.1 with the proofs given in the last section. This completes the proof of theorem 1.1 announced in the introduction. □

REMARK. The reader can check that the proof of theorem 5.1 can be adapted to a more general class of *suitable* words, namely words with several up and down clumps, which are not interleaved. However it can be proved that the corresponding notion of *amenable* words is exactly the same as that given above so there is no point in pursuing this.

REMARKS ON AMENABILITY

We believe that Klyachko's methods can, with further extension, be adapted to give a solution to the adjunction problem for torsion-free groups in general and we intend to pursue this in a later paper on the subject. However, as we have seen, his methods extend without too much work to the case of amenable words and we finish this section with a brief discussion of amenability, and in particular, consider how general is this class of amenable words.

In some sense (see below) nearly all t-sequences are amenable. However it is definitely *not* the case that *all* sequences are amenable. We now give some examples.

There are 17 t-sequences of lengths $\geqslant 2$ and $\leqslant 8$ (up to cyclic permutation, inversion and replacement of t by t^{-1}) which fail to be amenable while 30 are amenable. Examples of amenable t-sequences are $t^3t^{-1}tt^{-1}tt^{-1}$, $t^4t^{-2}tt^{-1}$, $t^2t^{-1}tt^{-2}tt^{-1}$ and examples of non-amenable sequences are t^8, $t^2t^{-2}t^2t^{-2}$, $t^3t^{-1}t^3t^{-1}$. Note that the 17 non-amenable sequences include several sequences for which the adjunction problem is solved, see below.

As length increases, the situation becomes progressively better and it can be checked that the proportion of non-amenable sequences tends to zero as length tends to ∞. Up to and including length 9, there is no difference between suitable and amenable sequences, but as length increases the difference becomes immense, with again very few sequences suitable compared with (nearly all) amenable. The first example of a t-sequence which is amenable but not suitable is $t^2t^{-1}t^2t^{-2}tt^{-2}$ and a more typical (longer) example is

$$t^3t^{-1}t^3t^{-2}tt^{-2}t^5t^{-4}.$$

The t-sequence t^n is interesting because the adjunction problem is already proved (without the torsion-free hypothesis) for words with this t-shape [L]. However the methods discussed here do not extend this result to t-sequences in normal form based on t^n. An example is $t^3t^{-1}t^3t^{-1}$.

Another interesting case is the sequence tt^{-1} which is not amenable. However a simple trick (substitute u^2 for t) makes it suitable. Hence theorem 5.1 implies a solution to the adjunction problem (over torsion-free groups) for words of the form $gtg't^{-1}$. For words of this shape, torsion-free is a necessary condition as the example in the introduction shows!

We do not yet have a simple test for amenability though it is easy from the definition to write down large classes of amenable sequences. However it can be seen that, speaking very roughly, a sequence is amenable unless it is has a uniform slope, like $t^5t^{-3}t^5t^{-3}$ or $t^3t^{-3}t^3t^{-3}$ (slope zero).

6. Further Applications

We give here the other applications from [Kl] of the crash theorems, not covered above.

Theorem 6.1 (Application to free products). *Let A, B be groups and suppose each (cyclic) factor of $u \in A * B - A$ has infinite order. Then the natural homomorphism $A \to \langle A * B \mid [A, u] = 1\rangle$ is injective.*

Proof. Suppose not. Then the conditions of the first transversality lemma apply and there is a non trivial element $a \in A$ such that $a \in \langle\langle [A, u]\rangle\rangle$. So we have a cell subdivision K of the 2-sphere such that reading round from the base point $*$ for every 2-cell in K spells out the word

$$w(a) = (c_0^{-1}ac_0)c_1 \cdots c_{n-1}(c_n^{-1}a^{-1}c_n)c_{n-1}^{-1} \cdots c_1^{-1}$$

for some $a \in A$, see figure 8. Note that if this 2-cell has the opposite orientation then the word spelt out is $w(a^{-1})$.

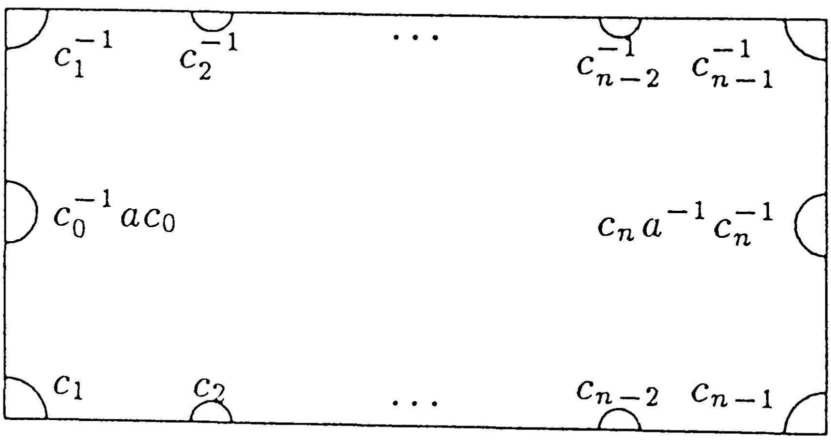

Figure 8
The 2-cell labelled by $w(a)$

Now consider the traffic flow defined as follows. The car associated to a 2-cell starts out from the base point $*$ and proceeds in an anticlockwise manner so that it takes a unit amount of time to reach the next corner. It is clear that any crash must take place at a 0-cell. By the crash theorem there are at least two total crashes so we can assume it takes place at a 0-cell where the angle labels multiply to 1. There are two cases to consider.

If the crash occurs at time 0 mod n then the clockwise labelling around this 0-cell is $c_\alpha^{-1}a_1c_\alpha, c_\alpha^{-1}a_2c_\alpha, \dots, c_\alpha^{-1}a_kc_\alpha$ where $a_1a_2\cdots a_k = 1$ and $\alpha = 0$ or $\alpha = n$. A simple calculation shows that the anticlockwise product of the remaining angles of these k 2-cells is 1. So we may simplify the situation by collapsing these k 2-cells to a point.

If the crash occurs at a time $\neq 0$ mod n then the clockwise labelling around this 0-cell is $c_i^k = 1$ for some $0 < i < n$ and some $k > 1$ contradicting the torsion free hypothesis. □

Let H, H' be groups and let $\phi: H \to H'$ be an isomorphism. We shall use the notation h^ϕ to denote the image of $h \in H$ under ϕ. Similarly we shall write $a^b := b^{-1}ab$ for conjugation.

THEOREM 6.2 (Application to HNN extensions). *Let H and H' be two isomorphic subgroups of the group A under the isomorphism $h \to h^\phi$, $h \in H$. Let B be a group and let $w \in A * B - A$ have torsion free factors. Then the natural map*

$$A \to \langle A, B \mid w^{-1}hw = h^\phi, h \in H\rangle$$

is injective.

Proof. Consider the following groups

$$\begin{aligned}
A' &= \langle A, t \mid t^{-1}ht = h^\phi, h \in H\rangle,\\
A'' &= \langle A, t, B \mid t^{-1}ht = h^\phi, [a, t^{-1}w] = 1, [t, w] = 1, h \in H, a \in A\rangle\\
&= \langle A', B \mid [a, t^{-1}w] = 1, [t, w] = 1, a \in A\rangle,\\
A''' &= \langle A, B \mid w^{-1}hw = h^\phi, h \in H\rangle.
\end{aligned}$$

We can construct the following commuting diagram,

$$\begin{tikzcd}
 & & A''' \arrow[d, "\gamma"] \\
A \arrow[r, "\alpha"] & A' \arrow[ur, "\delta"] \arrow[r, "\beta"] & A''
\end{tikzcd}$$

where the maps α, β, γ and δ are induced by inclusion. In order for γ to be a well defined homomorphism it is necessary to check that the relation $w^{-1}hw = h^\phi$, $h \in H$ is a consequence of the relations $t^{-1}ht = h^\phi$, $[a, t^{-1}w] = 1$, $[t, w] = 1$, $h \in H$, $a \in A$. But this follows because $w^{-1}hw = w^{-1}tt^{-1}htt^{-1}w = w^{-1}th^\phi t^{-1}w = h^\phi$. Now α is injective because A' is an HNN extension of A (see [DD, p. 33] or [Se, p. 9]) and β is injective because of theorem 6.1. So δ is injective and this proves the theorem. □

THEOREM 6.3. *Let*

$$(*) \qquad u_i(t) = 1,\ i \in I$$

be a set of equations over the group A *where the exponent sum of* t *in each* $u_i(t)$ *is zero. Suppose* $w = w(t) \in A * \langle t \rangle - A$ *and the factors of* w *are all torsion free. Then the set of equations*

$$(**) \qquad u_i(w(t)) = 1,\ i \in I$$

has a solution over A *if and only if the set* $(*)$ *has a solution over* A.

Proof. Let $w(t) = at$ where $a \in A$ has infinite order. Then a solution x for $u_i(w(t)) = 1$ defines a solution at for $(*)$.

Conversely suppose $x \in A'$ is a solution of the set of equations $\{u_i(t) = 1 \mid i \in I\}$. Let G be the subgroup of A' generated by

$$\{x^{-n}ax^n \mid a \in A, n \in \mathbf{Z}\}.$$

Then A is a subgroup of G and G is a subgroup of

$$H = \langle G, t \mid w^{-1}gw = g^\phi, g \in G\rangle$$

where $g^\phi = x^{-1}gx$ by theorem 6.2. Because of the exponent sum condition $u_i(w) = 1, i \in I$. □

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