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must descend to the Green's function of E^0. For the annulus $\tilde{G}$ may be constructed, for example, by the method of electrostatic images, using the reflections (4.3) in the boundary circles of A_1^{μ} (see [2], [7]).

The lemniscate (4.16) may serve as a useful model for domains with corners.

5. Involutions on a torus

We return to the situation at the beginning of section 3, but with a non-simply connected Riemann surface Γ. Let $\pi : \tilde{\Gamma} \to \Gamma$ be the universal covering space, and $\Lambda \subseteq Aut(\tilde{\Gamma})$ be the group of covering transformations. We consider liftings

$$\tilde{\tau}_i, \tilde{\rho} : \tilde{\Gamma} \to \tilde{\Gamma} \tag{5.1}$$

of τ_i, ρ. For each $\gamma \in \Lambda$ there is a $\gamma_1 \in \Lambda$ with

$$\tilde{\rho} \circ \gamma = \gamma_1 \circ \tilde{\rho} , \tag{5.2}$$

and similarly for τ_i. Also

$$\tilde{\tau}_i^2, \tilde{\rho}^2 \in \Lambda . \tag{5.3}$$

In this section we take $\tilde{\Gamma} = \mathbf{C}$, and Λ a group of translations, which we shall also identify with an additive subgroup of $(\mathbf{C}, +)$ of rank one or two over $\mathbf{Z}$. We shall determine what restrictions on Λ are forced if Γ is the complexification of a real curve admitting double valued reflection. We are, of course, interested in the corresponding objects on $\Gamma = \mathbf{C}/\Lambda$.

We drop the tilde notation and let $t \in \mathbf{C}$. In view of (5.3), we consider

$$\tau_i(t) = \varepsilon_i t + c_i, \varepsilon_i^2 = 1, (\varepsilon_i + 1)c_i \in \Lambda, i = 1, 2 ; \tag{5.4}$$

$$\sigma(t) = \tau_1 \tau_2(t) = \varepsilon_1 \varepsilon_2 t + c_1 + \varepsilon_1 c_2 ; \tag{5.5}$$

$$\rho(t) = a\bar{t} + b, a\bar{a} = 1, b + a\bar{b} \in \Lambda . \tag{5.6}$$

In case $\tau_2 = \rho \tau_1 \rho$, we have

$$\varepsilon_1 = \varepsilon_2, c_2 = a(\varepsilon_1 \bar{b} + \bar{c}_1) + b . \tag{5.7}$$

The constants c_i, b are only determined mod Λ. For each τ_i, either $\varepsilon_i = -1$ and $c_i \in \mathbf{C}$ can be arbitrary, or $\varepsilon_i = +1$ and $2c_i \in \Lambda$.

We set

$$a = e^{2\alpha i}, 0 \leqslant \alpha < \pi, \rho_\alpha(t) = a\bar{t} , \tag{5.8}$$
$$l_\alpha = \{\lambda e^{i\alpha} \mid \lambda \in \mathbf{R}\} = \{t \mid Re(ie^{-i\alpha} t) = 0\} .$$

ρ_α is the reflection the line l_α. If we apply the condition (3.3) we get

(5.9) $$\rho_\alpha(\Lambda) = \Lambda\ .$$

Thus, Λ must be symmetric about l_α. Clearly, $b + a\bar{b} \in l_\alpha$, so (5.6) gives

(5.10) $$b + a\bar{b} = \omega_0 \in \Lambda \cap l_\alpha\ ,$$

and b lies on the line perpendicular to l_α and passing through $\frac{1}{2}\omega_0$. This line has the equation

(5.11) $$2Re(e^{-i\alpha}(t - \omega_0/2)) = e^{-i\alpha}(t + a\bar{t} - \omega_0) = 0\ .$$

If Λ satisfies (5.9) for some angle α, we choose $\omega_0 \in \Lambda \cap l_\alpha$, for example $\omega_0 = 0$. We then choose b satisfying (5.10), and construct ρ. If we replace b by $b + \omega_*, \omega_* \in \Lambda \cap l_\alpha$, then ω_0 gets replaced by $\omega_0 + 2\omega_*$. Hence, there are at most two inequivalent choices for ω_0 on l_α.

A point $t_0 \in \mathbf{C}$ represents a fixed-point of ρ if and only if it lies on a line of the form

(5.12) $$t - a\bar{t} - b = \omega_0' \in \Lambda\ .$$

Since $t - a\bar{t}$ is orthogonal to $e^{i\alpha}$, ω_0' must lie on the line perpendicular to l_α and passing through $-\frac{1}{2}\omega_0$,

(5.13) $$t + a\bar{t} + \omega_0 = 0\ .$$

If there is an $\omega_0' \in \Lambda$ on this line, then the fixed-point set $FP(\rho)$ of ρ is non-empty, and is given by (5.12) for all such ω_0'. (5.12) is the line parallel to l_α and passing through $\frac{1}{2}(b + \omega_0')$; hence, there are at most two inequivalent choices of ω_0'.

First consider the very simple case

(5.14) $$\Lambda = \{2\pi k i \mid k \in \mathbf{Z}\}\ .$$

From (5.9) we can only have $\alpha = 0$, or $\alpha = \pi/2$. In the first case, l_α is the real axis, $a = 1$, and $\omega_0 = 0$, $b = ib_2$ is purely imaginary. We may take $\omega_0' = 2ki$, $k = 1, 2$; thus

(5.15) $$\rho(t) = \bar{t} + ib_2,\ FP(\rho) = \{Im\ t = b_2/2\} \cup \{Im\ t = b_2/2 + \pi\}\ .$$

In the second case l_α is the imaginary axis, $a = -1$, and we may take either $\omega_0 = 0$, or $\omega_0 = 2\pi i$. Then, either $b = b_1 \in \mathbf{R}$, or $b = b_1 + i\pi$. In the first case we have $\omega_0' = 0$, while in the second case there is no ω_0'. Thus,

(5.16) $$\rho(t) = -\bar{t} + b_1,\ FP(\rho) = \{Re\, t = b_1/2\}\ .$$

Of course, $\Gamma \equiv \mathbf{C}^*$, and the covering projection $\pi : \mathbf{C} \to \Gamma$ is just $\zeta = \pi(t) = e^t$. The first choice of ρ gives reflection in the two rays $\arg \zeta = \frac{1}{2} b_2, \frac{1}{2} b_2 + \pi$. The second gives reflection in the circle $|\zeta| = e^{b_1}$. We must still make a choice of τ_1 as in (5.4), and find a "minimal" function F which is τ_1-invariant. Relative to ζ we have $\tau_1(\zeta) = \mu \zeta^{\varepsilon_1}$, $\mu = e^{c_1}$. For $\varepsilon_1 = -1$, we take $F = f + f \circ \tau_1 = \zeta + \mu \zeta^{-1}$. For $\varepsilon_1 = +1$, $c_1 = \pi i$, we take $F = f \cdot f \circ \tau_1 = -\zeta^2$. We have already used these in the case of conics.

Next we consider a rank two lattice (4.8), and after a coordinate change if necessary, choose a normalized basis $\omega_1 = 1$, $\omega_2 = \omega$,

$$Im\, \omega > 0, \quad -\frac{1}{2} < Re\, \omega \leqslant \frac{1}{2}, \quad |\omega| \geqslant 1, \tag{5.17}$$

$$|\omega| = 1 \Rightarrow Re\, \omega \geqslant 0 .$$

We consider those Λ which satisfy the reality condition (5.9) [3], [5]. Since $a = \rho_\alpha(1) \in \Lambda$, we have $a = n_1 + n_2 \omega$, and

$$\begin{aligned} 1 = a\bar{a} &= n_1^2 + n_2^2 |\omega|^2 + 2 n_1 n_2 Re\, \omega \\ &\geqslant n_1^2 + n_2^2 - 2 |n_1 n_2 Re\, \omega| \\ &\geqslant |n_1|^2 + |n_2|^2 - |n_1 n_2| \geqslant |n_1 n_2| . \end{aligned} \tag{5.18}$$

There are two cases. If $n_1 n_2 = 0$, then either $a = \pm 1$, or $|\omega| = 1$ and $a = \pm \omega$. Otherwise, $|n_1| = |n_2| = 1$, and we have the equalities in (5.18). Equality in all three places implies $|\omega| = 1$, $Re\, \omega = \frac{1}{2}$, $n_1 n_2 \leqslant 0$, and $|n_1| = |n_2|$. Hence, $n_2 = -n_1 = \pm 1$, and $a = \pm(\omega - 1)$. $|\omega| = |\omega - 1| = 1$ implies that $\omega = (1 + \sqrt{3} i)/2$. If $a = \pm 1$, then both $\omega, \bar{\omega}$, and hence $2 Re\, \omega$ are in Λ. It follows that either $Re\, \omega = 0$, or $Re\, \omega = \frac{1}{2}$.

In summary we have the following classical result.

LEMMA 5.1. *Suppose that* $\mathbf{C}/\Lambda$ *admits the reflection (5.8). Then the possibilities for* Λ *and* a *are*

1. $Re\, \omega = 0, |\omega| > 1, a = \pm 1$;
2. $Re\, \omega = \frac{1}{2}, |\omega| > 1, a = \pm 1$;
3. $|\omega| = 1, 0 < Re\, \omega < \frac{1}{2}, a = \pm \omega$;
4. $\omega = i, a = \pm 1, \pm i$;
5. $\omega = (1 + \sqrt{3} i)/2, a = \pm 1, \pm \omega, \pm(\omega - 1)$.

In particular, it follows that $J(\omega)$, the elliptic modular function [5], is real at ω. In each case one has to determine the possible reflections ρ, determine their fixed-point sets, and add a suitable τ_1.

We consider the rectangular case (1) of the lemma, for application in the next section. Let

(5.19) $$\omega_1 = 1, \omega_2 = \omega = i\omega'', \omega'' > 1$$

be a normalized basis. For $a = 1$, l_α is the real axis, $\omega_0 = 0$, or $\omega_0 = 1$, $b = ib_2$, or $b = \frac{1}{2} + ib_2$, $0 \leqslant b_2 < \omega''$. In the first case $\omega_0' = 0$, or $\omega_0' = \omega$, while there is no ω_0' in the second case. Thus, we have

(5.20) $$\rho(t) = \bar{t} + ib_2, \; FP(\rho) = \{Im\, t = b_2/2\} \cup \{Im\, t = (b_2 + \omega'')/2\}.$$

For $a = -1$, l_α is the imaginary axis, $\omega_0 = 0$ or $\omega_0 = \omega$, $b = b_1$, or $b = b_1 + i\omega''/2$, $0 \leqslant b_1 < 1$. $\omega_0' = 0, 1$ in the first case, and there is no ω_0' in the second case. We have

(5.21) $$\rho(t) = -\bar{t} + b_1, \; FP(\rho) = \{Re\, t = b_1/2\} \cup \{Re\, t = (b_1 + 1)/2\}.$$

If $\varepsilon_1 = -1$, then

(5.22) $$FP(\tau_1) = \{c_1/2, (c_1 + \omega_1)/2, (c_1 + \omega_2)/2, (c_1 + \omega_1 + \omega_2)/2\}.$$

If we have $\varepsilon_1 = +1$, $2c_1 \in \Lambda$, $c_1 \notin \Lambda$, then τ_1 has no fixed points. τ_1 is then the deck transformation of an unbranched covering of another torus.

6. Embedding of tori

We turn to the problem of concretely realizing the data of the previous section in the main case. Given a complex torus $\Gamma = \mathbf{C}/\Lambda$, with a pair of holomorphic involutions induced by

(6.1) $$\tau_i(t) = -t + c_i, \; i = 1, 2,$$

we look for a pair of two-fold branched coverings

(6.2) $$\pi_i : \Gamma \to \mathbf{P}_1, \; \pi_i \circ \tau_i = \pi_i, \; i = 1, 2.$$

The problem is immediately solved by taking

(6.3) $$z_i = \pi_i(t) \equiv \mathscr{P}(t - c_i/2), \; i = 1, 2,$$