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must descend to the Green's function of  $E^0$ . For the annulus  $\tilde{G}$  may be constructed, for example, by the method of electrostatic images, using the reflections (4.3) in the boundary circles of  $A_1^\mu$  (see [2], [7]).

The lemniscate (4.16) may serve as a useful model for domains with corners.

## 5. INVOLUTIONS ON A TORUS

We return to the situation at the beginning of section 3, but with a non-simply connected Riemann surface  $\Gamma$ . Let  $\pi: \tilde{\Gamma} \rightarrow \Gamma$  be the universal covering space, and  $\Lambda \subseteq \text{Aut}(\tilde{\Gamma})$  be the group of covering transformations. We consider liftings

$$(5.1) \quad \tilde{\tau}_i, \tilde{\rho}: \tilde{\Gamma} \rightarrow \tilde{\Gamma}$$

of  $\tau_i, \rho$ . For each  $\gamma \in \Lambda$  there is a  $\gamma_1 \in \Lambda$  with

$$(5.2) \quad \tilde{\rho} \circ \gamma = \gamma_1 \circ \tilde{\rho},$$

and similarly for  $\tau_i$ . Also

$$(5.3) \quad \tilde{\tau}_i^2, \tilde{\rho}^2 \in \Lambda.$$

In this section we take  $\tilde{\Gamma} = \mathbf{C}$ , and  $\Lambda$  a group of translations, which we shall also identify with an additive subgroup of  $(\mathbf{C}, +)$  of rank one or two over  $\mathbf{Z}$ . We shall determine what restrictions on  $\Lambda$  are forced if  $\Gamma$  is the complexification of a real curve admitting double valued reflection. We are, of course, interested in the corresponding objects on  $\Gamma = \mathbf{C}/\Lambda$ .

We drop the tilde notation and let  $t \in \mathbf{C}$ . In view of (5.3), we consider

$$(5.4) \quad \tau_i(t) = \varepsilon_i t + c_i, \varepsilon_i^2 = 1, (\varepsilon_i + 1)c_i \in \Lambda, i = 1, 2;$$

$$(5.5) \quad \sigma(t) = \tau_1 \tau_2(t) = \varepsilon_1 \varepsilon_2 t + c_1 + \varepsilon_1 c_2;$$

$$(5.6) \quad \rho(t) = a\bar{t} + b, a\bar{a} = 1, b + a\bar{b} \in \Lambda.$$

In case  $\tau_2 = \rho \tau_1 \rho$ , we have

$$(5.7) \quad \varepsilon_1 = \varepsilon_2, c_2 = a(\varepsilon_1 \bar{b} + \bar{c}_1) + b.$$

The constants  $c_i, b$  are only determined mod  $\Lambda$ . For each  $\tau_i$ , either  $\varepsilon_i = -1$  and  $c_i \in \mathbf{C}$  can be arbitrary, or  $\varepsilon_i = +1$  and  $2c_i \in \Lambda$ .

We set

$$(5.8) \quad a = e^{2\alpha i}, 0 \leq \alpha < \pi, \rho_\alpha(t) = a\bar{t},$$

$$l_\alpha = \{\lambda e^{ia} \mid \lambda \in \mathbf{R}\} = \{t \mid \text{Re}(ie^{-i\alpha} t) = 0\}.$$

$\rho_\alpha$  is the reflection the line  $l_\alpha$ . If we apply the condition (3.3) we get

$$(5.9) \quad \rho_\alpha(\Lambda) = \Lambda .$$

Thus,  $\Lambda$  must be symmetric about  $l_\alpha$ . Clearly,  $b + a\bar{b} \in l_\alpha$ , so (5.6) gives

$$(5.10) \quad b + a\bar{b} = \omega_0 \in \Lambda \cap l_\alpha ,$$

and  $b$  lies on the line perpendicular to  $l_\alpha$  and passing through  $\frac{1}{2}\omega_0$ . This line has the equation

$$(5.11) \quad 2Re(e^{-i\alpha}(t - \omega_0/2)) = e^{-i\alpha}(t + a\bar{t} - \omega_0) = 0 .$$

If  $\Lambda$  satisfies (5.9) for some angle  $\alpha$ , we choose  $\omega_0 \in \Lambda \cap l_\alpha$ , for example  $\omega_0 = 0$ . We then choose  $b$  satisfying (5.10), and construct  $\rho$ . If we replace  $b$  by  $b + \omega_*$ ,  $\omega_* \in \Lambda \cap l_\alpha$ , then  $\omega_0$  gets replaced by  $\omega_0 + 2\omega_*$ . Hence, there are at most two inequivalent choices for  $\omega_0$  on  $l_\alpha$ .

A point  $t_0 \in \mathbf{C}$  represents a fixed-point of  $\rho$  if and only if it lies on a line of the form

$$(5.12) \quad t - a\bar{t} - b = \omega'_0 \in \Lambda .$$

Since  $t - a\bar{t}$  is orthogonal to  $e^{i\alpha}$ ,  $\omega'_0$  must lie on the line perpendicular to  $l_\alpha$  and passing through  $-\frac{1}{2}\omega_0$ ,

$$(5.13) \quad t + a\bar{t} + \omega_0 = 0 .$$

If there is an  $\omega'_0 \in \Lambda$  on this line, then the fixed-point set  $FP(\rho)$  of  $\rho$  is non-empty, and is given by (5.12) for all such  $\omega'_0$ . (5.12) is the line parallel to  $l_\alpha$  and passing through  $\frac{1}{2}(b + \omega'_0)$ ; hence, there are at most two inequivalent choices of  $\omega'_0$ .

First consider the very simple case

$$(5.14) \quad \Lambda = \{2\pi k i \mid k \in \mathbf{Z}\} .$$

From (5.9) we can only have  $\alpha = 0$ , or  $\alpha = \pi/2$ . In the first case,  $l_\alpha$  is the real axis,  $a = 1$ , and  $\omega_0 = 0$ ,  $b = ib_2$  is purely imaginary. We may take  $\omega'_0 = 2ki$ ,  $k = 1, 2$ ; thus

$$(5.15) \quad \rho(t) = \bar{t} + ib_2, \quad FP(\rho) = \{Im t = b_2/2\} \cup \{Im t = b_2/2 + \pi\} .$$

In the second case  $l_\alpha$  is the imaginary axis,  $a = -1$ , and we may take either  $\omega_0 = 0$ , or  $\omega_0 = 2\pi i$ . Then, either  $b = b_1 \in \mathbf{R}$ , or  $b = b_1 + i\pi$ . In the first case we have  $\omega'_0 = 0$ , while in the second case there is no  $\omega'_0$ . Thus,

$$(5.16) \quad \rho(t) = -\bar{t} + b_1, \quad FP(\rho) = \{Re t = b_1/2\} .$$

Of course,  $\Gamma \equiv \mathbf{C}^*$ , and the covering projection  $\pi: \mathbf{C} \rightarrow \Gamma$  is just  $\zeta = \pi(t) = e^t$ . The first choice of  $\rho$  gives reflection in the two rays  $\arg \zeta = \frac{1}{2}b_2, \frac{1}{2}b_2 + \pi$ . The second gives reflection in the circle  $|\zeta| = e^{b_1}$ . We must still make a choice of  $\tau_1$  as in (5.4), and find a “minimal” function  $F$  which is  $\tau_1$ -invariant. Relative to  $\zeta$  we have  $\tau_1(\zeta) = \mu \zeta^{\varepsilon_1}$ ,  $\mu = e^{c_1}$ . For  $\varepsilon_1 = -1$ , we take  $F = f + f \circ \tau_1 = \zeta + \mu \zeta^{-1}$ . For  $\varepsilon_1 = +1$ ,  $c_1 = \pi i$ , we take  $F = f \cdot f \circ \tau_1 = -\zeta^2$ . We have already used these in the case of conics.

Next we consider a rank two lattice (4.8), and after a coordinate change if necessary, choose a normalized basis  $\omega_1 = 1$ ,  $\omega_2 = \omega$ ,

$$(5.17) \quad \text{Im } \omega > 0, \quad -\frac{1}{2} < \text{Re } \omega \leq \frac{1}{2}, \quad |\omega| \geq 1,$$

$$|\omega| = 1 \Rightarrow \text{Re } \omega \geq 0.$$

We consider those  $\Lambda$  which satisfy the reality condition (5.9) [3], [5]. Since  $a = \rho_a(1) \in \Lambda$ , we have  $a = n_1 + n_2 \omega$ , and

$$(5.18) \quad \begin{aligned} 1 = a\bar{a} &= n_1^2 + n_2^2 |\omega|^2 + 2n_1 n_2 \text{Re } \omega \\ &\geq n_1^2 + n_2^2 - 2 |n_1 n_2 \text{Re } \omega| \\ &\geq |n_1|^2 + |n_2|^2 - |n_1 n_2| \geq |n_1 n_2|. \end{aligned}$$

There are two cases. If  $n_1 n_2 = 0$ , then either  $a = \pm 1$ , or  $|\omega| = 1$  and  $a = \pm \omega$ . Otherwise,  $|n_1| = |n_2| = 1$ , and we have the equalities in (5.18). Equality in all three places implies  $|\omega| = 1$ ,  $\text{Re } \omega = \frac{1}{2}$ ,  $n_1 n_2 \leq 0$ , and  $|n_1| = |n_2|$ . Hence,  $n_2 = -n_1 = \pm 1$ , and  $a = \pm(\omega - 1)$ .  $|\omega| = |\omega - 1| = 1$  implies that  $\omega = (1 + \sqrt{3}i)/2$ . If  $a = \pm 1$ , then both  $\omega, \bar{\omega}$ , and hence  $2\text{Re } \omega$  are in  $\Lambda$ . It follows that either  $\text{Re } \omega = 0$ , or  $\text{Re } \omega = \frac{1}{2}$ .

In summary we have the following classical result.

LEMMA 5.1. *Suppose that  $\mathbf{C}/\Lambda$  admits the reflection (5.8). Then the possibilities for  $\Lambda$  and  $a$  are*

1.  $\text{Re } \omega = 0, |\omega| > 1, a = \pm 1$ ;
2.  $\text{Re } \omega = \frac{1}{2}, |\omega| > 1, a = \pm 1$ ;
3.  $|\omega| = 1, 0 < \text{Re } \omega < \frac{1}{2}, a = \pm \omega$ ;
4.  $\omega = i, a = \pm 1, \pm i$ ;
5.  $\omega = (1 + \sqrt{3}i)/2, a = \pm 1, \pm \omega, \pm(\omega - 1)$ .

In particular, it follows that  $J(\omega)$ , the elliptic modular function [5], is real at  $\omega$ . In each case one has to determine the possible reflections  $\rho$ , determine their fixed-point sets, and add a suitable  $\tau_1$ .

We consider the rectangular case (1) of the lemma, for application in the next section. Let

$$(5.19) \quad \omega_1 = 1, \omega_2 = \omega = i\omega'', \omega'' > 1$$

be a normalized basis. For  $a = 1$ ,  $l_a$  is the real axis,  $\omega_0 = 0$ , or  $\omega_0 = 1$ ,  $b = ib_2$ , or  $b = \frac{1}{2} + ib_2$ ,  $0 \leq b_2 < \omega''$ . In the first case  $\omega'_0 = 0$ , or  $\omega'_0 = \omega$ , while there is no  $\omega'_0$  in the second case. Thus, we have

$$(5.20) \quad \rho(t) = \bar{t} + ib_2, \quad FP(\rho) = \{Im t = b_2/2\} \cup \{Im t = (b_2 + \omega'')/2\}.$$

For  $a = -1$ ,  $l_a$  is the imaginary axis,  $\omega_0 = 0$  or  $\omega_0 = \omega$ ,  $b = b_1$ , or  $b = b_1 + i\omega''/2$ ,  $0 \leq b_1 < 1$ .  $\omega'_0 = 0, 1$  in the first case, and there is no  $\omega'_0$  in the second case. We have

$$(5.21) \quad \rho(t) = -\bar{t} + b_1, \quad FP(\rho) = \{Re t = b_1/2\} \cup \{Re t = (b_1 + 1)/2\}.$$

If  $\varepsilon_1 = -1$ , then

$$(5.22) \quad FP(\tau_1) = \{c_1/2, (c_1 + \omega_1)/2, (c_1 + \omega_2)/2, (c_1 + \omega_1 + \omega_2)/2\}.$$

If we have  $\varepsilon_1 = +1$ ,  $2c_1 \in \Lambda$ ,  $c_1 \notin \Lambda$ , then  $\tau_1$  has no fixed points.  $\tau_1$  is then the deck transformation of an unbranched covering of another torus.

## 6. EMBEDDING OF TORI

We turn to the problem of concretely realizing the data of the previous section in the main case. Given a complex torus  $\Gamma = \mathbf{C}/\Lambda$ , with a pair of holomorphic involutions induced by

$$(6.1) \quad \tau_i(t) = -t + c_i, \quad i = 1, 2,$$

we look for a pair of two-fold branched coverings

$$(6.2) \quad \pi_i: \Gamma \rightarrow \mathbf{P}_1, \quad \pi_i \circ \tau_i = \pi_i, \quad i = 1, 2.$$

The problem is immediately solved by taking

$$(6.3) \quad z_i = \pi_i(t) \equiv \mathcal{P}(t - c_i/2), \quad i = 1, 2,$$