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must descend to the Green's function of  $E^0$ . For the annulus  $\tilde{G}$  may be constructed, for example, by the method of electrostatic images, using the reflections (4.3) in the boundary circles of  $A_1^{\mu}$  (see [2], [7]).

The lemniscate (4.16) may serve as a useful model for domains with corners.

# 5. INVOLUTIONS ON A TORUS

We return to the situation at the beginning of section 3, but with a non-simply connected Riemann surface  $\Gamma$ . Let  $\pi: \tilde{\Gamma} \to \Gamma$  be the universal covering space, and  $\Lambda \subseteq Aut(\tilde{\Gamma})$  be the group of covering transformations. We consider liftings

(5.1) 
$$\tilde{\tau}_i, \tilde{\rho}: \Gamma \to \Gamma$$

of  $\tau_i$ ,  $\rho$ . For each  $\gamma \in \Lambda$  there is a  $\gamma_1 \in \Lambda$  with

(5.2) 
$$\tilde{\rho} \circ \gamma = \gamma_1 \circ \tilde{\rho}$$
,

and similarly for  $\tau_i$ . Also

(5.3) 
$$\tilde{\tau}_i^2, \tilde{\rho}^2 \in \Lambda$$

In this section we take  $\tilde{\Gamma} = \mathbf{C}$ , and  $\Lambda$  a group of translations, which we shall also identify with an additive subgroup of  $(\mathbf{C}, +)$  of rank one or two over  $\mathbf{Z}$ . We shall determine what restrictions on  $\Lambda$  are forced if  $\Gamma$  is the complexification of a real curve admitting double valued reflection. We are, of course, interested in the corresponding objects on  $\Gamma = \mathbf{C}/\Lambda$ .

We drop the tilde notation and let  $t \in \mathbb{C}$ . In view of (5.3), we consider

(5.4) 
$$\tau_i(t) = \varepsilon_i t + c_i, \varepsilon_i^2 = 1, (\varepsilon_i + 1) c_i \in \Lambda, i = 1, 2;$$

(5.5) 
$$\sigma(t) = \tau_1 \tau_2(t) = \varepsilon_1 \varepsilon_2 t + c_1 + \varepsilon_1 c_2;$$

(5.6) 
$$\rho(t) = a\bar{t} + b, a\bar{\alpha} = 1, b + a\bar{b} \in \Lambda.$$

In case  $\tau_2 = \rho \tau_1 \rho$ , we have

(5.7) 
$$\varepsilon_1 = \varepsilon_2, c_2 = a(\varepsilon_1 \overline{b} + \overline{c_1}) + b$$

The constants  $c_i$ , b are only determined mod  $\Lambda$ . For each  $\tau_i$ , either  $\varepsilon_i = -1$  and  $c_i \in \mathbb{C}$  can be arbitrary, or  $\varepsilon_i = +1$  and  $2c_i \in \Lambda$ . We set

(5.8)  $a = e^{2\alpha i}, \ 0 \leq \alpha < \pi, \ \rho_{\alpha}(t) = a\overline{t},$  $l_{\alpha} = \{\lambda e^{i\alpha} \mid \lambda \in \mathbf{R}\} = \{t \mid Re(ie^{-i\alpha}t) = 0\}.$ 

 $\rho_{\alpha}$  is the reflection the line  $l_{\alpha}$ . If we apply the condition (3.3) we get

$$\rho_{\alpha}(\Lambda) = \Lambda$$

Thus,  $\Lambda$  must be symmetric about  $l_{\alpha}$ . Clearly,  $b + a\bar{b} \in l_{\alpha}$ , so (5.6) gives

$$(5.10) b + ab = \omega_0 \in \Lambda \cap l_{\alpha},$$

and b lies on the line perpendicular to  $l_{\alpha}$  and passing through  $\frac{1}{2}\omega_0$ . This line has the equation

(5.11) 
$$2Re(e^{-i\alpha}(t-\omega_0/2)) = e^{-i\alpha}(t+a\bar{t}-\omega_0) = 0.$$

If  $\Lambda$  satisfies (5.9) for some angle  $\alpha$ , we choose  $\omega_0 \in \Lambda \cap l_{\alpha}$ , for example  $\omega_0 = 0$ . We then choose *b* satisfying (5.10), and construct  $\rho$ . If we replace *b* by  $b + \omega_*, \omega_* \in \Lambda \cap l_{\alpha}$ , then  $\omega_0$  gets replaced by  $\omega_0 + 2\omega_*$ . Hence, there are at most two inequivalent choices for  $\omega_0$  on  $l_{\alpha}$ .

A point  $t_0 \in \mathbb{C}$  represents a fixed-point of  $\rho$  if and only if it lies on a line of the form

(5.12) 
$$t - a\overline{t} - b = \omega'_0 \in \Lambda .$$

Since  $t - a\bar{t}$  is orthogonal to  $e^{i\alpha}$ ,  $\omega'_0$  must lie on the line perpendicular to  $l_{\alpha}$  and passing through  $-\frac{1}{2}\omega_0$ ,

$$(5.13) t + a\bar{t} + \omega_0 = 0 .$$

If there is an  $\omega'_0 \in \Lambda$  on this line, then the fixed-point set  $FP(\rho)$  of  $\rho$  is non-empty, and is given by (5.12) for all such  $\omega'_0$ . (5.12) is the line parallel to  $l_{\alpha}$  and passing through  $\frac{1}{2}(b + \omega'_0)$ ; hence, there are at most two inequivalent choices of  $\omega'_0$ .

First consider the very simple case

(5.14) 
$$\Lambda = \{2\pi ki \mid k \in \mathbb{Z}\}.$$

From (5.9) we can only have  $\alpha = 0$ , or  $\alpha = \pi/2$ . In the first case,  $l_{\alpha}$  is the real axis, a = 1, and  $\omega_0 = 0$ ,  $b = ib_2$  is purely imaginary. We may take  $\omega'_0 = 2ki$ , k = 1, 2; thus

(5.15) 
$$\rho(t) = \overline{t} + ib_2, \ FP(\rho) = \{Im \ t = b_2/2\} \cup \{Im \ t = b_2/2 + \pi\}.$$

In the second case  $l_{\alpha}$  is the imaginary axis, a = -1, and we may take either  $\omega_0 = 0$ , or  $\omega_0 = 2\pi i$ . Then, either  $b = b_1 \in \mathbf{R}$ , or  $b = b_1 + i\pi$ . In the first case we have  $\omega'_0 = 0$ , while in the second case there is no  $\omega'_0$ . Thus,

(5.16)  $\rho(t) = -\bar{t} + b_1, FP(\rho) = \{Re\,t = b_1/2\}.$ 

Of course,  $\Gamma \equiv \mathbb{C}^*$ , and the covering projection  $\pi: \mathbb{C} \to \Gamma$  is just  $\zeta = \pi(t) = e^t$ . The first choice of  $\rho$  gives reflection in the two rays  $\arg \zeta = \frac{1}{2}b_2, \frac{1}{2}b_2 + \pi$ . The second gives reflection in the circle  $|\zeta| = e^{b_1}$ . We must still make a choice of  $\tau_1$  as in (5.4), and find a "minimal" function F which is  $\tau_1$ -invariant. Relative to  $\zeta$  we have  $\tau_1(\zeta) = \mu \zeta^{\varepsilon_1}$ ,  $\mu = e^{c_1}$ . For  $\varepsilon_1 = -1$ , we take  $F = f + f \circ \tau_1 = \zeta + \mu \zeta^{-1}$ . For  $\varepsilon_1 = +1$ ,  $c_1 = \pi i$ , we take  $F = f \cdot f \circ \tau_1 = -\zeta^2$ . We have already used these in the case of conics.

Next we consider a rank two lattice (4.8), and after a coordinate change if necessary, choose a normalized basis  $\omega_1 = 1$ ,  $\omega_2 = \omega$ ,

(5.17) 
$$Im \ \omega > 0, \quad -\frac{1}{2} < Re \ \omega \leq \frac{1}{2}, \quad | \ \omega | \ge 1,$$
$$| \ \omega | = 1 \Rightarrow Re \ \omega \ge 0.$$

We consider those  $\Lambda$  which satisfy the reality condition (5.9) [3], [5]. Since  $a = \rho_{\alpha}(1) \in \Lambda$ , we have  $a = n_1 + n_2\omega$ , and

(5.18)  

$$1 = a\bar{a} = n_1^2 + n_2^2 |\omega|^2 + 2n_1n_2 Re\omega$$

$$\geqslant n_1^2 + n_2^2 - 2 |n_1n_2 Re\omega|$$

$$\geqslant |n_1|^2 + |n_2|^2 - |n_1n_2| \geqslant |n_1n_2|$$

There are two cases. If  $n_1n_2 = 0$ , then either  $a = \pm 1$ , or  $|\omega| = 1$  and  $a = \pm \omega$ . Otherwise,  $|n_1| = |n_2| = 1$ , and we have the equalities in (5.18). Equality in all three places implies  $|\omega| = 1$ ,  $Re \omega = \frac{1}{2}$ ,  $n_1n_2 \leq 0$ , and  $|n_1| = |n_2|$ . Hence,  $n_2 = -n_1 = \pm 1$ , and  $a = \pm (\omega - 1)$ .  $|\omega| = |\omega - 1| = 1$  implies that  $\omega = (1 + \sqrt{3}i)/2$ . If  $a = \pm 1$ , then both  $\omega, \bar{\omega}$ , and hence  $2Re \omega$  are in  $\Lambda$ . It follows that either  $Re \omega = 0$ , or  $Re \omega = \frac{1}{2}$ .

In summary we have the following classical result.

LEMMA 5.1. Suppose that  $\mathbb{C}/\Lambda$  admits the reflection (5.8). Then the possibilities for  $\Lambda$  and a are

1. 
$$Re \omega = 0, |\omega| > 1, a = \pm 1;$$
  
2.  $Re \omega = \frac{1}{2}, |\omega| > 1, a = \pm 1;$   
3.  $|\omega| = 1, 0 < Re \omega < \frac{1}{2}, a = \pm \omega;$   
4.  $\omega = i, a = \pm 1, \pm i;$   
5.  $\omega = (1 + \sqrt{3}i)/2, a = \pm 1, \pm \omega, \pm (\omega - 1)$ 

In particular, it follows that  $J(\omega)$ , the elliptic modular function [5], is real at  $\omega$ . In each case one has to determine the possible reflections  $\rho$ , determine their fixed-point sets, and add a suitable  $\tau_1$ .

We consider the rectangular case (1) of the lemma, for application in the next section. Let

(5.19) 
$$\omega_1 = 1, \, \omega_2 = \omega = i\omega^{\prime\prime}, \, \omega^{\prime\prime} > 1$$

be a normalized basis. For a = 1,  $l_{\alpha}$  is the real axis,  $\omega_0 = 0$ , or  $\omega_0 = 1$ ,  $b = ib_2$ , or  $b = \frac{1}{2} + ib_2$ ,  $0 \le b_2 < \omega''$ . In the first case  $\omega'_0 = 0$ , or  $\omega'_0 = \omega$ , while there is no  $\omega'_0$  in the second case. Thus, we have

(5.20) 
$$\rho(t) = \overline{t} + ib_2, \ FP(\rho) = \{Im \ t = b_2/2\} \cup \{Im \ t = (b_2 + \omega'')/2\}$$

For a = -1,  $l_{\alpha}$  is the imaginary axis,  $\omega_0 = 0$  or  $\omega_0 = \omega$ ,  $b = b_1$ , or  $b = b_1 + i\omega''/2$ ,  $0 \le b_1 < 1$ .  $\omega'_0 = 0, 1$  in the first case, and there is no  $\omega'_0$  in the second case. We have

(5.21) 
$$\rho(t) = -\bar{t} + b_1, \ FP(\rho) = \{Re\,t = b_1/2\} \cup \{Re\,t = (b_1 + 1)/2\}.$$

If 
$$\varepsilon_1 = -1$$
, then

(5.22) 
$$FP(\tau_1) = \{c_1/2, (c_1 + \omega_1)/2, (c_1 + \omega_2)/2, (c_1 + \omega_1 + \omega_2)/2\}.$$

If we have  $\varepsilon_1 = +1$ ,  $2c_1 \in \Lambda$ ,  $c_1 \notin \Lambda$ , then  $\tau_1$  has no fixed points.  $\tau_1$  is then the deck transformation of an unbranched covering of another torus.

## 6. EMBEDDING OF TORI

We turn to the problem of concretely realizing the data of the previous section in the main case. Given a complex torus  $\Gamma = \mathbf{C}/\Lambda$ , with a pair of holomorphic involutions induced by

(6.1) 
$$\tau_i(t) = -t + c_i, i = 1, 2,$$

we look for a pair of two-fold branched coverings

(6.2) 
$$\pi_i \colon \Gamma \to \mathbf{P}_1, \ \pi_i \circ \tau_i = \pi_i, \ i = 1, 2.$$

The problem is immediately solved by taking

(6.3) 
$$z_i = \pi_i(t) \equiv \mathscr{P}(t - c_i/2), \ i = 1, 2,$$