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Adding and subtracting to eliminate t gives

$$r(z, \bar{w}) \equiv (z - \bar{w})^2 - 16\alpha(z + \bar{w}) + 64\alpha^2 = 0\,, \tag{3.21}$$

which is (2.18) with $\alpha = 4a$.

REMARK. In the above examples we chose the simplest non-trivial rational functions $f(t)$, which led us back to the examples of section 2. Other choices of f would lead to more complicated rational curves.

4. RIEMANN MAPS

The deeper geometric and analytic properties of a simply connected proper subdomain $D \subset \mathbf{C}$ are brought out in the problem of mapping it conformally onto the unit disc Δ, or right half plane H. In this section we shall indicate by example what role double valued reflection plays in this problem.

Thus, let the boundary ∂D be a branch of a real algebraic curve admitting double valued reflection. The Riemann map, $f: D \to \Delta$, continues to some neighborhood of the closure $\bar{D}$, and so maps a curve with double valued reflection to one with single valued reflection. This forces f to possess additional symmetry properties. Roughly speaking, if f could be continued globally, then the two reflected points of any point z would have to map to the single reflected point of $f(z)$. This is decisive in determining an explicit expression for f.

We first consider the domain D inside the ellipse (2.2). The first map, $z = \pi_1(t)$, in (3.14) takes the annulus $A_1^{\mu} = \{1 < |t| < \mu\}$ onto D, as a two fold covering

$$\pi_1 : A_1^{\mu} \to D\,, \tag{4.1}$$

branched at the points $t = \pm\lambda \in A_1^{\mu}$. We have

$$\pi^{-1}(\gamma) = \partial A_1^{\mu} = \gamma_1 \cup \gamma_{\mu}\,, \tag{4.2}$$

where γ_1 is the fixed point set of ρ, and $\gamma_{\mu} = \tau_1(\gamma_1)$ is the fixed point set of $\rho_{\mu} = \tau_1 \rho \tau_1$,

$$\rho(t) = 1/\bar{t}, \quad \rho_{\mu}(t) = \mu/\bar{t}\,. \tag{4.3}$$

The Riemann map,

$$f: D \to H, \zeta = f(z)\,, \tag{4.4}$$

will, of course, be as given by H. A. Schwarz [10], [9], except that we choose to map to the right half plane $H: Re\,\zeta > 0$, rather than to the unit disc. We assume that f maps the vertex $-a_1$ to 0, and the vertex $+a_1$ to ∞. It follows that $h \equiv f \circ \pi_1$ will have simple zeros at $\pi_1^{-1}(-a_1)$ and simple poles at $\pi_1^{-1}(a_1)$. Since h is purely imaginary on the boundary of A_1^{μ}, we can extend it to successively larger and larger annuli by the two reflections

$$h = \hat{\rho} \circ h \circ \rho, \quad h = \hat{\rho} \circ h \circ \rho_{\mu}, \quad \hat{\rho}(\zeta) = -\bar{\zeta}\,. \tag{4.5}$$

It follows by (3.1) that the extended function h must satisfy

$$h = h \circ \rho_{\mu} \circ \rho = h \circ (\tau_1 \rho \tau_1) \circ \rho = h \circ \sigma\,. \tag{4.6}$$

This extended function h must also remain invariant under τ_1 by analytic continuation of functional relations. Equivalently, h is invariant under both τ_1 and τ_2. Hence, we seek $h(t)$ meromorphic for $0 < |t| < \infty$, satisfying

$$h \circ \sigma(t) = h(\mu^2 t) = h(t),$$
$$h \circ \tau_1(t) = h(\mu/t) = h(t)\,,$$

and having simple zeros at $t = -1$, $-\mu$, and simple poles at $t = +1$, $+\mu$, $(\mu > 1)$.

We set

$$t = e^s, \quad \varphi(s) = h(e^s)\,. \tag{4.7}$$

Then φ is to be doubly periodic with respect to the lattice

$$\Lambda = \{n_1\omega_1 + n_2\omega_2 \mid n_1, n_2 \in \mathbf{Z}\}\,, \tag{4.8}$$

where

$$\omega_1 = 2\log\mu > 0, \quad \omega_2 = 2\pi i\,. \tag{4.9}$$

It is to have simple zeros at points congruent, mod Λ, to

$$a_1 = 2\pi i, \quad a_2 = \omega_1/2\,, \tag{4.10}$$

and simple poles at points congruent to

$$b_1 = \pi i, \quad b_2 = \pi i + \omega_1/2\,. \tag{4.11}$$

Since $a_1 + a_2 = b_1 + b_2$, φ can be represented as the Weierstrass Sigma quotient [5], [6]

$$\varphi(s) = c\,\frac{S(s-a_1)\,S(s-a_2)}{S(s-b_1)\,S(s-b_2)}\,, \tag{4.12}$$

where

$$S(s) = s \prod_{\omega \in \Lambda - \{0\}} \left(1 - \frac{s}{\omega}\right) \exp\left(\frac{s}{\omega} + \frac{1}{2}\left(\frac{s}{\omega}\right)^2\right) . \tag{4.13}$$

The map τ_1 is covered by the map

$$\tilde{\tau}_1(s) = -s + \omega_1/2 . \tag{4.14}$$

φ and $\varphi \circ \tilde{\tau}_1$ have poles and zeros at the same points, with the same orders. Hence, $\varphi \circ \tilde{\tau}_1/\varphi = c_1$, where $c_1^2 = 1$, since τ_1 is an involution. Since the sum of the residues at the two poles of each is zero, one can see, using the form (4.14), that $c_1 = +1$. Hence, φ is automatically $\tilde{\tau}_1$-invariant. We have proved the following equivalent of the theorem of Schwarz [10].

THEOREM 4.1. *The Riemann map (4.4) of the ellipse D onto the right half plane H has the form*

$$f(z) = \varphi(\log(z \pm \sqrt{z^2 - \mu})) , \tag{4.15}$$

where φ is given by (4.12), (4.13).

As another example we consider the conformal map f from the domain D to the right of the right branch of the hyperbola (2.1) onto the right half plane H. That this problem is more "unstable" than the previous one may be seen by making the inversion $z \mapsto 1/z$. The hyperbola goes into the lemniscate

$$Bz\bar{z} - A(z^2 + \bar{z}^2) = z^2\bar{z}^2 , \tag{4.16}$$

and D goes into one of the bounded domains $\tilde{D}$ which (4.16) bounds. $\tilde{D}$ has a corner at 0 with angle ε,

$$\tan \varepsilon = \sqrt{\frac{B - 2A}{-B - 2A}} . \tag{4.17}$$

The mapping problem is rather sensitive to the rationality properties of ε relative to π. The two branches of the lemniscate at 0 lift to different sheets of the branched covering $z = \pi_1(t)$, (3.19), the whole curve being the image of the real t-axis.

π_1 maps $t > 0$ onto the right branch of the hyperbola, i.e. onto ∂D. The sector

$$S_0^{2\alpha} = \{0 < \arg t < 2\alpha = \arg \mu\}, \lambda = e^{i\alpha} \tag{4.18}$$

is mapped 2-to-1 onto D, with the branch point λ going to the focus $z = a$. If we commence to extend $h \equiv f \circ \pi_1$ by reflecting in the sides of $S_0^{2\alpha}$, we are likely to get a multiple valued map. Hence, we set

$$t = e^{2\alpha s/\pi},\ s = s_1 + is_2,\ \varphi(s) = h(e^{2\alpha s/\pi})\ , \tag{4.19}$$

so that $0 < s_2 < \pi$ is mapped to $S_0^{2\alpha}$. The maps σ and τ_1 are covered by

$$\tilde{\sigma}(s) = s + 2\pi i,\ \tilde{\tau}_1(s) = -s + \pi i\ . \tag{4.20}$$

Thus, φ is $2\pi i$-periodic, purely imaginary for $Im\ s = 0, \pi$, and $\varphi(-s + \pi i) = \varphi(s)$. The function $\varphi(s) = \sin(is)$ satisfies these conditions. Thus, [9]

$$f(z) = \sin\left(\frac{\pi i}{2\alpha}\log[\lambda(z \pm \sqrt{z^2 - 1})]\right)\ . \tag{4.21}$$

If $\frac{2\alpha}{\pi} = \frac{p}{q}$ is rational, then one can avoid the transcendental functions in (4.21). We set

$$t = s^{p/q},\ \varphi(s) = h(s^{p/q})\ . \tag{4.22}$$

Then φ reflects across the real axis and has simple zeros at $s = \pm 1$ and simple poles at $s = 0, \infty$. Thus, $\varphi(s) = ci(s - s^{-1})$, and we get

$$f(z) = ci[(\lambda(z \pm \sqrt{z^2 - 1}))^{q/p} - (\lambda(z \pm \sqrt{z^2 - 1}))^{p/q}]\ . \tag{4.23}$$

All the above maps are, of course, well known. The point here is that they follow naturally from our theory, as also does the Riemann map of the inside of a parabola using (3.20). One might hope to "explain" all such explicit maps within the current framework.

In place of the Riemann map we may consider the Green's function. We briefly consider the case of the ellipse D. Let $\tilde{G}(t, t_0)$ be the Green's function for A_1^μ, with pole at t_0 . We have

$$\tilde{G}(\tau_1(t), \tau_1(t_0)) = \tilde{G}(t, t_0)\ , \tag{4.24}$$

since τ_1 is an *involutive* automorphism of A_1^μ. It follows that

$$\begin{aligned} G(t, t_0) &= \frac{1}{2}[\tilde{G}(t, t_0) + \tilde{G}(t, \tau_1(t_0))] \\ &= \frac{1}{2}[\tilde{G}(t, t_0) + \tilde{G}(\tau_1(t), t_0)] \end{aligned} \tag{4.25}$$

must descend to the Green's function of E^0. For the annulus $\tilde{G}$ may be constructed, for example, by the method of electrostatic images, using the reflections (4.3) in the boundary circles of A_1^μ (see [2], [7]).

The lemniscate (4.16) may serve as a useful model for domains with corners.

5. Involutions on a torus

We return to the situation at the beginning of section 3, but with a non-simply connected Riemann surface Γ. Let $\pi : \tilde{\Gamma} \to \Gamma$ be the universal covering space, and $\Lambda \subseteq Aut(\tilde{\Gamma})$ be the group of covering transformations. We consider liftings

$$\tilde{\tau}_i, \tilde{\rho} : \tilde{\Gamma} \to \tilde{\Gamma} \tag{5.1}$$

of τ_i, ρ. For each $\gamma \in \Lambda$ there is a $\gamma_1 \in \Lambda$ with

$$\tilde{\rho} \circ \gamma = \gamma_1 \circ \tilde{\rho}\,, \tag{5.2}$$

and similarly for τ_i. Also

$$\tilde{\tau}_i^2, \tilde{\rho}^2 \in \Lambda\,. \tag{5.3}$$

In this section we take $\tilde{\Gamma} = \mathbf{C}$, and Λ a group of translations, which we shall also identify with an additive subgroup of $(\mathbf{C}, +)$ of rank one or two over $\mathbf{Z}$. We shall determine what restrictions on Λ are forced if Γ is the complexification of a real curve admitting double valued reflection. We are, of course, interested in the corresponding objects on $\Gamma = \mathbf{C}/\Lambda$.

We drop the tilde notation and let $t \in \mathbf{C}$. In view of (5.3), we consider

$$\tau_i(t) = \varepsilon_i t + c_i, \varepsilon_i^2 = 1, (\varepsilon_i + 1)c_i \in \Lambda, i = 1, 2\,; \tag{5.4}$$

$$\sigma(t) = \tau_1\tau_2(t) = \varepsilon_1\varepsilon_2 t + c_1 + \varepsilon_1 c_2\,; \tag{5.5}$$

$$\rho(t) = a\bar{t} + b, a\bar{a} = 1, b + a\bar{b} \in \Lambda\,. \tag{5.6}$$

In case $\tau_2 = \rho\tau_1\rho$, we have

$$\varepsilon_1 = \varepsilon_2, c_2 = a(\varepsilon_1\bar{b} + \bar{c}_1) + b\,. \tag{5.7}$$

The constants c_i, b are only determined mod Λ. For each τ_i, either $\varepsilon_i = -1$ and $c_i \in \mathbf{C}$ can be arbitrary, or $\varepsilon_i = +1$ and $2c_i \in \Lambda$.

We set

$$a = e^{2\alpha i}, 0 \leqslant \alpha < \pi, \rho_\alpha(t) = a\bar{t}\,, \tag{5.8}$$
$$l_\alpha = \{\lambda e^{i\alpha} \mid \lambda \in \mathbf{R}\} = \{t \mid Re(ie^{-i\alpha}t) = 0\}\,.$$