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## 3. THE INTRINSIC THEORY: GENUS ZERO

From the intrinsic point of view we start with a pair of holomorphic involutions  $\tau_i: \Gamma \rightarrow \Gamma$ ,  $i = 1, 2$ , on an abstract Riemann surface  $\Gamma$ . The quotient spaces  $\Gamma/\tau_i \equiv \Gamma_i$  have natural analytic structures [4], and  $\tau_i$  is the covering involution for the branched covering  $\pi_i: \Gamma \rightarrow \Gamma_i$ . If

$$(3.1) \quad \tau_2 = \rho \tau_1 \rho$$

for an anti-holomorphic involution  $\rho$  on  $\Gamma$ , then there exists an anti-biholomorphic map  $\hat{\rho}: \Gamma_1 \rightarrow \Gamma_2$  with  $\hat{\rho} \circ \pi_1 = \pi_2 \circ \rho$ . We are mainly concerned with the case  $\Gamma_1 = \Gamma_2 \subseteq \mathbf{P}_1$ , although one could study real analytic curves on an arbitrary Riemann surface  $\Gamma_1$ . If  $\Gamma$  is compact, and  $\Gamma_1 = \mathbf{P}_1$ , then  $\Gamma$  is hyperelliptic. The existence of the two functionally independent 2-fold branched coverings  $\pi_i: \Gamma \rightarrow \mathbf{P}_1$  forces  $\Gamma$  to be either an elliptic or rational curve [4]. We shall restrict to these two cases, in this paper.

In the genus zero case,  $\Gamma = \mathbf{P}_1$ , which we consider in this section, the holomorphic involutions are fractional linear maps. A single one  $\tau(t)$  can be normalized so that its fixed points are  $t = 0, \infty$ , and hence has the form  $\tau(t) = -t$ . The theory of a pair of such involutions is still elementary, but somewhat involved, so we shall refer to [8] for some details.

For a pair of holomorphic involutions  $\tau_1, \tau_2$ , let the fixed-point sets be

$$(3.2) \quad FP(\tau_i) = \{p_i, q_i\}, \quad i = 1, 2.$$

If  $\tau_1$  and  $\tau_2$  have the same fixed-point sets, they are equal. They have a single common fixed point in the parabolic case. We first consider the general case in which the four points  $\{p_1, q_1, p_2, q_2\}$  are all distinct. We may form their cross ratio,

$$(3.3) \quad \kappa = \frac{(p_1 - p_2)(q_1 - q_2)}{(p_1 - q_2)(q_1 - p_2)}.$$

Interchanging  $\tau_1$  and  $\tau_2$ , or  $p_1$  with  $q_1$ , or  $p_2$  with  $q_2$  results in (at most) the change  $\kappa \mapsto 1/\kappa$ . Thus, the conditions  $\kappa > 0$ ,  $\kappa < 0$ ,  $Re \kappa = 0$ ,  $\kappa \bar{\kappa} = 1$ , for example, are intrinsic conditions on the pair  $\tau_i$ . The first two occur when  $\tau_1$  and  $\tau_2$  are intertwined by an anti-holomorphic involution  $\rho$ . The significance of the second two conditions is still rather mysterious at this point.

The maps  $\tau_1, \tau_2$  may be represented in homogeneous coordinates  $(\xi, \eta) \in \mathbf{C}^2$  for  $\mathbf{P}_1$  by a pair of linear involutions. As in section 2 of [8] they may be chosen as follows,

$$(3.4) \quad \begin{aligned} \tau_1(\xi, \eta) &= (\lambda\eta, \lambda^{-1}\xi), \quad \tau_2(\xi, \eta) = (\lambda^{-1}\eta, \lambda\xi), \\ \sigma(\xi, \eta) &= (\mu\xi, \mu^{-1}\eta), \quad \mu = \lambda^2. \end{aligned}$$

In the non-homogeneous coordinate  $t = \xi/\eta$ ,

$$(3.5) \quad \tau_1(t) = \frac{\mu}{t}, \quad \tau_2(t) = \frac{1}{\mu t}, \quad \sigma(t) = \mu^2 t.$$

Since

$$(3.6) \quad FP(\tau_1) = \{\lambda, -\lambda\}, \quad FP(\tau_2) = \{\lambda^{-1}, -\lambda^{-1}\},$$

we have

$$(3.7) \quad \kappa = \left( \frac{1 - \mu}{1 + \mu} \right)^2.$$

An anti-holomorphic involution  $\rho$  of  $\mathbf{P}_1$  is given by reflection in some circle, which is anti-linear in homogeneous coordinates. Thus, lemma 2.2 of [8] applies directly to give the following.

LEMMA 3.1. *The normal form for the triple  $\tau_1, \tau_2, \rho$ , with  $\tau_2\rho = \rho\tau_1$ , falls into two cases. The  $\tau_i$  are still given by (3.4) or (3.5), while*

$$(3.8) \quad \lambda = \bar{\lambda} > 1, \quad \rho(\xi, \eta) = (\bar{\eta}, \bar{\xi}), \quad \rho(t) = 1/\bar{t},$$

or

$$(3.9) \quad \lambda\bar{\lambda} = 1, \quad 0 < \arg \lambda < \pi/2, \quad \rho(\xi, \eta) = (\bar{\xi}, \bar{\eta}), \quad \rho(t) = \bar{t}.$$

(3.11) is the elliptic case with  $\kappa > 0$ . (3.12) is the hyperbolic case, where  $\kappa < 0$ .

Next we consider the problem of realizing the data  $\tau_i$  by means of an analytic curve,

$$(3.10) \quad z = \pi_1(t), \quad \bar{w} = \pi_2(t), \quad \pi_i \circ \tau_i = \pi_i.$$

This amounts to finding suitable functions  $\pi_i$  invariant under  $\tau_i$ . We shall also impose the reality condition

$$(3.11) \quad \bar{\pi}_2 = \pi_1 \circ \rho.$$

In general we can try  $\pi_i = f + f \circ \tau_i$ , for any analytic or meromorphic function  $f$ . Taking  $f(t) = t$  leads to the "Zhukovsky functions",

$$(3.12) \quad z = \frac{\alpha}{2} \left( t + \frac{\mu}{t} \right), \quad \bar{w} = \frac{\beta}{2} \left( t + \frac{1}{\mu t} \right),$$

where  $\alpha, \beta$  are constants. Computing  $z^2, \bar{w}^2, z\bar{w}$ , and eliminating  $t$  leads to the equation

$$(3.13) \quad \frac{4}{\alpha\beta} \left( \mu + \frac{1}{\mu} \right) z\bar{w} - 4 \left( \frac{1}{\mu\alpha^2} z^2 + \frac{\mu}{\beta^2} \bar{w}^2 \right) = \left( \mu - \frac{1}{\mu} \right)^2.$$

Next we choose the constants so that (3.11) holds. For the case (3.8) we take  $\bar{\beta} = \alpha\mu$ ,  $\alpha = 1$ , so that

$$(3.14) \quad z = \frac{1}{2} \left( t + \frac{\mu}{t} \right), \quad \bar{w} = \frac{\mu}{2} \left( t + \frac{1}{\mu t} \right),$$

and (3.13) becomes (2.6) with

$$(3.15) \quad B = \frac{4(1 + \mu^2)}{(1 - \mu^2)^2}, \quad A = \frac{4\mu}{(1 - \mu^2)^2}, \quad B - 2A = \frac{4}{1 + \mu^2}.$$

Since the last two numbers are positive, we have an ellipse with foci on the real axis.

For the case (3.9) we choose  $\beta = \bar{\alpha}$ , and  $\alpha = \bar{\lambda}$ , so that the coefficients of  $z^2$  and  $\bar{w}^2$  in (3.16) are equal. We get

$$(3.16) \quad z = \frac{\bar{\lambda}}{2} \left( t + \frac{\mu}{t} \right), \quad \bar{w} = \frac{\lambda}{2} \left( t + \frac{1}{\mu t} \right),$$

and equation (2.6) with

$$(3.17) \quad B = \frac{4(\mu + \bar{\mu})}{(\mu - \bar{\mu})^2}, \quad A = \frac{4}{(\mu - \bar{\mu})^2}, \quad B - 2A = \frac{4(\mu + \bar{\mu} - 2)}{(\mu - \bar{\mu})^2}.$$

It follows that  $A < 0$ , and  $B - 2A > 0$ , since  $-2 < \mu + \bar{\mu} < 2$ , by (3.9). Thus we have a hyperbola with foci on the real axis.

In the parabolic case we may assume that  $q_1 = q_2 = \infty$ , and  $p_1 = 1$ ,  $p_2 = -1$ . Then

$$(3.18) \quad \tau_1(t) = -t + 2, \quad \tau_2(t) = -t - 2.$$

If we take

$$(3.19) \quad \rho(t) = -\bar{t},$$

then  $\tau_2 = \rho\tau_1\rho$ . We can satisfy (3.13) and (3.14) if we take  $\pi_1 = f + f \circ \tau_1$ , where  $f \circ \rho = \bar{f}$ . Thus we take  $f(t) = \alpha t^2$ ,  $\alpha = \bar{\alpha}$ ,

$$(3.20) \quad z = 2\alpha(t - 1)^2, \quad \bar{w} = 2\alpha(t + 1)^2.$$

Adding and subtracting to eliminate  $t$  gives

$$(3.21) \quad r(z, \bar{w}) \equiv (z - \bar{w})^2 - 16\alpha(z + \bar{w}) + 64\alpha^2 = 0,$$

which is (2.18) with  $\alpha = 4a$ .

REMARK. In the above examples we chose the simplest non-trivial rational functions  $f(t)$ , which led us back to the examples of section 2. Other choices of  $f$  would lead to more complicated rational curves.

#### 4. RIEMANN MAPS

The deeper geometric and analytic properties of a simply connected proper subdomain  $D \subset \mathbb{C}$  are brought out in the problem of mapping it conformally onto the unit disc  $\Delta$ , or right half plane  $H$ . In this section we shall indicate by example what role double valued reflection plays in this problem.

Thus, let the boundary  $\partial D$  be a branch of a real algebraic curve admitting double valued reflection. The Riemann map,  $f: D \rightarrow \Delta$ , continues to some neighborhood of the closure  $\bar{D}$ , and so maps a curve with double valued reflection to one with single valued reflection. This forces  $f$  to possess additional symmetry properties. Roughly speaking, if  $f$  could be continued globally, then the two reflected points of any point  $z$  would have to map to the single reflected point of  $f(z)$ . This is decisive in determining an explicit expression for  $f$ .

We first consider the domain  $D$  inside the ellipse (2.2). The first map,  $z = \pi_1(t)$ , in (3.14) takes the annulus  $A_1^\mu = \{1 < |t| < \mu\}$  onto  $D$ , as a two fold covering

$$(4.1) \quad \pi_1: A_1^\mu \rightarrow D,$$

branched at the points  $t = \pm \lambda \in A_1^\mu$ . We have

$$(4.2) \quad \pi^{-1}(\gamma) = \partial A_1^\mu = \gamma_1 \cup \gamma_\mu,$$

where  $\gamma_1$  is the fixed point set of  $\rho$ , and  $\gamma_\mu = \tau_1(\gamma_1)$  is the fixed point set of  $\rho_\mu = \tau_1 \rho \tau_1$ ,

$$(4.3) \quad \rho(t) = 1/\bar{t}, \quad \rho_\mu(t) = \mu/\bar{t}.$$

The Riemann map,

$$(4.4) \quad f: D \rightarrow H, \quad \zeta = f(z),$$