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3. THE INTRINSIC THEORY: GENUS ZERO

From the intrinsic point of view we start with a pair of holomorphic involutions $\tau_i: \Gamma \rightarrow \Gamma$, $i = 1, 2$, on an abstract Riemann surface Γ . The quotient spaces $\Gamma/\tau_i \equiv \Gamma_i$ have natural analytic structures [4], and τ_i is the covering involution for the branched covering $\pi_i: \Gamma \rightarrow \Gamma_i$. If

$$(3.1) \quad \tau_2 = \rho \tau_1 \rho$$

for an anti-holomorphic involution ρ on Γ , then there exists an anti-biholomorphic map $\hat{\rho}: \Gamma_1 \rightarrow \Gamma_2$ with $\hat{\rho} \circ \pi_1 = \pi_2 \circ \rho$. We are mainly concerned with the case $\Gamma_1 = \Gamma_2 \subseteq \mathbf{P}_1$, although one could study real analytic curves on an arbitrary Riemann surface Γ_1 . If Γ is compact, and $\Gamma_1 = \mathbf{P}_1$, then Γ is hyperelliptic. The existence of the two functionally independent 2-fold branched coverings $\pi_i: \Gamma \rightarrow \mathbf{P}_1$ forces Γ to be either an elliptic or rational curve [4]. We shall restrict to these two cases, in this paper.

In the genus zero case, $\Gamma = \mathbf{P}_1$, which we consider in this section, the holomorphic involutions are fractional linear maps. A single one $\tau(t)$ can be normalized so that its fixed points are $t = 0, \infty$, and hence has the form $\tau(t) = -t$. The theory of a pair of such involutions is still elementary, but somewhat involved, so we shall refer to [8] for some details.

For a pair of holomorphic involutions τ_1, τ_2 , let the fixed-point sets be

$$(3.2) \quad FP(\tau_i) = \{p_i, q_i\}, \quad i = 1, 2.$$

If τ_1 and τ_2 have the same fixed-point sets, they are equal. They have a single common fixed point in the parabolic case. We first consider the general case in which the four points $\{p_1, q_1, p_2, q_2\}$ are all distinct. We may form their cross ratio,

$$(3.3) \quad \kappa = \frac{(p_1 - p_2)(q_1 - q_2)}{(p_1 - q_2)(q_1 - p_2)}.$$

Interchanging τ_1 and τ_2 , or p_1 with q_1 , or p_2 with q_2 results in (at most) the change $\kappa \mapsto 1/\kappa$. Thus, the conditions $\kappa > 0$, $\kappa < 0$, $Re \kappa = 0$, $\kappa \bar{\kappa} = 1$, for example, are intrinsic conditions on the pair τ_i . The first two occur when τ_1 and τ_2 are intertwined by an anti-holomorphic involution ρ . The significance of the second two conditions is still rather mysterious at this point.

The maps τ_1, τ_2 may be represented in homogeneous coordinates $(\xi, \eta) \in \mathbf{C}^2$ for \mathbf{P}_1 by a pair of linear involutions. As in section 2 of [8] they may be chosen as follows,

$$(3.4) \quad \begin{aligned} \tau_1(\xi, \eta) &= (\lambda\eta, \lambda^{-1}\xi), \quad \tau_2(\xi, \eta) = (\lambda^{-1}\eta, \lambda\xi), \\ \sigma(\xi, \eta) &= (\mu\xi, \mu^{-1}\eta), \quad \mu = \lambda^2. \end{aligned}$$

In the non-homogeneous coordinate $t = \xi/\eta$,

$$(3.5) \quad \tau_1(t) = \frac{\mu}{t}, \quad \tau_2(t) = \frac{1}{\mu t}, \quad \sigma(t) = \mu^2 t.$$

Since

$$(3.6) \quad FP(\tau_1) = \{\lambda, -\lambda\}, \quad FP(\tau_2) = \{\lambda^{-1}, -\lambda^{-1}\},$$

we have

$$(3.7) \quad \kappa = \left(\frac{1 - \mu}{1 + \mu} \right)^2.$$

An anti-holomorphic involution ρ of \mathbf{P}_1 is given by reflection in some circle, which is anti-linear in homogeneous coordinates. Thus, lemma 2.2 of [8] applies directly to give the following.

LEMMA 3.1. *The normal form for the triple τ_1, τ_2, ρ , with $\tau_2\rho = \rho\tau_1$, falls into two cases. The τ_i are still given by (3.4) or (3.5), while*

$$(3.8) \quad \lambda = \bar{\lambda} > 1, \quad \rho(\xi, \eta) = (\bar{\eta}, \bar{\xi}), \quad \rho(t) = 1/\bar{t},$$

or

$$(3.9) \quad \lambda\bar{\lambda} = 1, \quad 0 < \arg \lambda < \pi/2, \quad \rho(\xi, \eta) = (\bar{\xi}, \bar{\eta}), \quad \rho(t) = \bar{t}.$$

(3.11) is the elliptic case with $\kappa > 0$. (3.12) is the hyperbolic case, where $\kappa < 0$.

Next we consider the problem of realizing the data τ_i by means of an analytic curve,

$$(3.10) \quad z = \pi_1(t), \quad \bar{w} = \pi_2(t), \quad \pi_i \circ \tau_i = \pi_i.$$

This amounts to finding suitable functions π_i invariant under τ_i . We shall also impose the reality condition

$$(3.11) \quad \bar{\pi}_2 = \pi_1 \circ \rho.$$

In general we can try $\pi_i = f + f \circ \tau_i$, for any analytic or meromorphic function f . Taking $f(t) = t$ leads to the "Zhukovsky functions",

$$(3.12) \quad z = \frac{\alpha}{2} \left(t + \frac{\mu}{t} \right), \quad \bar{w} = \frac{\beta}{2} \left(t + \frac{1}{\mu t} \right),$$

where α, β are constants. Computing $z^2, \bar{w}^2, z\bar{w}$, and eliminating t leads to the equation

$$(3.13) \quad \frac{4}{\alpha\beta} \left(\mu + \frac{1}{\mu} \right) z\bar{w} - 4 \left(\frac{1}{\mu\alpha^2} z^2 + \frac{\mu}{\beta^2} \bar{w}^2 \right) = \left(\mu - \frac{1}{\mu} \right)^2.$$

Next we choose the constants so that (3.11) holds. For the case (3.8) we take $\bar{\beta} = \alpha\mu$, $\alpha = 1$, so that

$$(3.14) \quad z = \frac{1}{2} \left(t + \frac{\mu}{t} \right), \quad \bar{w} = \frac{\mu}{2} \left(t + \frac{1}{\mu t} \right),$$

and (3.13) becomes (2.6) with

$$(3.15) \quad B = \frac{4(1 + \mu^2)}{(1 - \mu^2)^2}, \quad A = \frac{4\mu}{(1 - \mu^2)^2}, \quad B - 2A = \frac{4}{1 + \mu^2}.$$

Since the last two numbers are positive, we have an ellipse with foci on the real axis.

For the case (3.9) we choose $\beta = \bar{\alpha}$, and $\alpha = \bar{\lambda}$, so that the coefficients of z^2 and \bar{w}^2 in (3.16) are equal. We get

$$(3.16) \quad z = \frac{\bar{\lambda}}{2} \left(t + \frac{\mu}{t} \right), \quad \bar{w} = \frac{\lambda}{2} \left(t + \frac{1}{\mu t} \right),$$

and equation (2.6) with

$$(3.17) \quad B = \frac{4(\mu + \bar{\mu})}{(\mu - \bar{\mu})^2}, \quad A = \frac{4}{(\mu - \bar{\mu})^2}, \quad B - 2A = \frac{4(\mu + \bar{\mu} - 2)}{(\mu - \bar{\mu})^2}.$$

It follows that $A < 0$, and $B - 2A > 0$, since $-2 < \mu + \bar{\mu} < 2$, by (3.9). Thus we have a hyperbola with foci on the real axis.

In the parabolic case we may assume that $q_1 = q_2 = \infty$, and $p_1 = 1$, $p_2 = -1$. Then

$$(3.18) \quad \tau_1(t) = -t + 2, \quad \tau_2(t) = -t - 2.$$

If we take

$$(3.19) \quad \rho(t) = -\bar{t},$$

then $\tau_2 = \rho\tau_1\rho$. We can satisfy (3.13) and (3.14) if we take $\pi_1 = f + f \circ \tau_1$, where $f \circ \rho = \bar{f}$. Thus we take $f(t) = \alpha t^2$, $\alpha = \bar{\alpha}$,

$$(3.20) \quad z = 2\alpha(t - 1)^2, \quad \bar{w} = 2\alpha(t + 1)^2.$$

Adding and subtracting to eliminate t gives

$$(3.21) \quad r(z, \bar{w}) \equiv (z - \bar{w})^2 - 16\alpha(z + \bar{w}) + 64\alpha^2 = 0 ,$$

which is (2.18) with $\alpha = 4a$.

REMARK. In the above examples we chose the simplest non-trivial rational functions $f(t)$, which led us back to the examples of section 2. Other choices of f would lead to more complicated rational curves.

4. RIEMANN MAPS

The deeper geometric and analytic properties of a simply connected proper subdomain $D \subset \mathbb{C}$ are brought out in the problem of mapping it conformally onto the unit disc Δ , or right half plane H . In this section we shall indicate by example what role double valued reflection plays in this problem.

Thus, let the boundary ∂D be a branch of a real algebraic curve admitting double valued reflection. The Riemann map, $f: D \rightarrow \Delta$, continues to some neighborhood of the closure \bar{D} , and so maps a curve with double valued reflection to one with single valued reflection. This forces f to possess additional symmetry properties. Roughly speaking, if f could be continued globally, then the two reflected points of any point z would have to map to the single reflected point of $f(z)$. This is decisive in determining an explicit expression for f .

We first consider the domain D inside the ellipse (2.2). The first map, $z = \pi_1(t)$, in (3.14) takes the annulus $A_1^\mu = \{1 < |t| < \mu\}$ onto D , as a two fold covering

$$(4.1) \quad \pi_1: A_1^\mu \rightarrow D ,$$

branched at the points $t = \pm \lambda \in A_1^\mu$. We have

$$(4.2) \quad \pi^{-1}(\gamma) = \partial A_1^\mu = \gamma_1 \cup \gamma_\mu ,$$

where γ_1 is the fixed point set of ρ , and $\gamma_\mu = \tau_1(\gamma_1)$ is the fixed point set of $\rho_\mu = \tau_1 \rho \tau_1$,

$$(4.3) \quad \rho(t) = 1/\bar{t}, \quad \rho_\mu(t) = \mu/\bar{t} .$$

The Riemann map,

$$(4.4) \quad f: D \rightarrow H, \quad \zeta = f(z) ,$$