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Kapitel: Can we teach students to "think mathematically" ?

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such as these resort to damage limitation using rote-learning of procedures as reported in another investigation:

> ...everyone is faced with courses whose purpose they have failed to grasp, let alone their finer details. Faced with this problem, most people set about finding typical questions and memorising the typical answers. Many gain excellent marks in courses of which they have no knowledge. *(Second year university mathematics student)*

What else can the failing student do? As Freudenthal said succinctly:

> ...the only thing the pupil can do with the ready-made mathematics which he is offered is to reproduce it. *(Freudenthal, 1973, p. 117)*

CAN WE TEACH STUDENTS TO "THINK MATHEMATICALLY"?

Can we encourage students to think like mathematicians? Even though we may not make every student a budding research mathematician, can we not alter attitudes and methods of doing mathematics that fosters a *creative* way of learning?

If students are given a suitable environment to relax and think about problems of an appropriate level, then such aspirations prove to be easy to attain. Typical problems (to be found in *Thinking Mathematically*, Mason *et al.,* 1982) include:

- If a square is cut into regions by straight lines, how many colours are needed so that no two adjoining regions are painted the same colour?
- Into how many squares can one cut a square?

These problems, on the face of it fairly easy, prove to be challenging, especially when *proof* is required — for instance proving that it is *not* possible to cut a square into two, three, or five squares. The latter statement proves to be true under certain circumstances, but false under others. I will not spoil it by refining the conditions on the problem, except to say that the alternative solution was given by a thirteen year old girl in a master class, when it had not occurred to me or to several hundred mathematics undergraduates over a decade of problem-solving classes.

Reflective thinking in mathematics is built up by Mason *et al.,* following the *How To Solve It* approach of Pólya (1945), but made more student-friendly by breaking problem-solving into three phases. The first is an *entry* phase in which the student must focus on the nature of the problem by asking "what do I want", reflect on any knowledge that may be available to begin the attack ("what do I know") and then think "what can I introduce" to move from what is known to what is wanted. The second phase is an *attack* which occurs when sufficient information is at hand to start to make the connections, and leads either to a dead-end, or to an insight which moves the problem on. The "dead-end" is seen as a valuable state because at least one method tried has not worked and by returning to the entry phase and re-assessing the position a new attack may ensue. If an insight occurs which may appear to solve the problem, then the third *review* phase needs to be undertaken, checking the method carefully, reflecting on how it was achieved and storing away strategies for future problems, then considering how to extend the problem in new ways. Whatever level of student participating in such a course, be it with children in school or final year mathematics students, the result appears always to be the same — a release from the routines of learning mathematics to pass examinations and a new spirit of adventure and confidence bred from success.

Yudariah Binte Mohammad Yusof worked with me as I taught the course on one occasion and she developed attitudinal questionnaires to ask students their opinions about mathematics and problem-solving. She then taught the course herself at another university and questioned 44 of her students before and after the course and then six months later during which time they took regular mathematics courses again (Mohd Yusof & Tall, 1994, 1995). She also showed the questionnaires to 22 lecturers who taught the students various courses and asked them "How do you *expect* a typical student to respond ?", then "How would you *prefer* the students to respond ?"

In almost every case the change in student response from before to after the problem-solving course proved to be in the same direction as the change from what the lecturers expected to what they preferred. Thus the *problem-solving caused an attitudinal change in the students in the direction desired by the lecturers.* However, in almost every case, *during the regular mathematics the students' attitudes turned back again towards what the lecturers expected and away from what they desired.*

Some students appreciated that their knowledge in problem solving helped them to learn mathematics and solve problems more effectively :

> The problem solving techniques help me come to terms with the abstract nature of the maths I am doing. I try to connect the [mathematical] ideas together and talk about it with my friends. It is not that easy though. But I felt all the effort worth it when I am able to do so. *(Male industrial science student, majoring in mathematics)*

But a considerable minority (14 out of 44, that is 32%) reported that the mathematics they were being taught did not allow them to think in a problem-solving manner:

> Since following the course I know mathematics is about solving problems. But whatever mathematics I am doing now doesn't allow me to do all those things. They are just more things to be remembered.
>
> *(Male computer education student)*

> I believed mathematics is useful in that it helps me to think. Having said that it is hard to say how I can do this with the maths I am doing. Most of the questions given can be solved by applying directly the procedures we had just learned. There is nothing to think about.
>
> *(Female industrial science student majoring in mathematics)*

So what does this tell us? One interpretation may be that the problem-solving had relatively easy problems that allowed the students to "think mathematically" but that "serious" mathematics is demanding.

Evidence from another source suggests that more open methods can work in analysis courses. In an experiment in Grenoble, large classes of analysis students were encouraged to work in groups in the lecture hall to propose theorems which they and other students subjected to a process of either proof or refutation by counter-example. A small minority said they preferred being told how to do mathematics in lecture classes, but 80% said they preferred the exploratory form of learning (Alibert, 1988; Alibert & Thomas, 1991).

It seems to me more likely that, because we fear failure in our students, we resort to the methods that "seem" necessary throughout mathematics. When students are likely to fail, we lack the faith in their ability to think for themselves and *tell* them how to do the mathematics in an organised way. The result is that they behave as we expect, rather than as we might prefer — they learn the material to pass the exam.

REFLECTIONS ON MATHEMATICAL THINKING

Currently the university mathematics community is under some stress because it earns part of its finance from teaching undergraduates and all is not well. In the UK the London Mathematical Society produced a report which changed the British undergraduate degree structure to allow for four years instead of the traditional three. Yet when I asked the LMS to change my area of research interest to "Advanced Mathematical Thinking", the committee reluctantly refused because it was not an accepted heading in the American Mathematical Society's listing of topics. A formal request passed to the AMS