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enormous value teaching science students who have little time for the formal niceties. It proves a good foundation for mathematics majors too, but one must not underestimate the difficulties of linking the visual imagery — which comes as a simultaneous whole — and the logical proofs which involve a different kind of sequential thinking.

USING SYMBOLISM TO COMPRESS PROCESS INTO CONCEPT

Symbols such as $Ax = c$ for a system of linear equations express a relationship in a far more compact form than any corresponding use of natural language. But there is a common use of symbols in mathematics which introduces compression in a subtle way rarely used in ordinary language. It is a method of compression that mathematicians are aware of intuitively but do not articulate in any formal sense, yet it becomes of vital importance in cognitive development. Let me illustrate this with the concept of number and the difference between a mathematician's definition and the cognitive development of the concept.

According to the set-theoretic view of Bourbaki, (cardinal) number concepts are about equivalences between sets. But a set-theoretic approach to number was tried in the "new math" of the sixties and it failed. Why ? Almost certainly because the set-theoretic approach is a natural systematisation when everything has been constructed and organised but it is less suitable as the beginning of a *cognitive* development. In essence it is a formulation which is likely to be suggested by experts who have forgotten their earlier development (cognitive principle I) but it proves unsuitable as an approach for the growing individual.

Even though small numbers of two or three objects can be recognised in a glance, cardinal numbers for these and larger numbers begin cognitively in young children as a *process* : the process of counting. Only later do the number symbols become recognised as manipulable number *concepts*.

It often happens that a mathematical process (such as counting) is symbolised, then the symbol is treated as a mathematical concept and itself manipulated as a mental object. Here are just a few examples :

symbol	*process*	*concept*
$3+2$	addition	sum
-3	subtract 3, 3 steps left	negative 3
$3/4$	division	fraction
$3+2x$	evaluation	expression
$v=s/t$	ratio	rate
$\sin A = \dfrac{opposite}{hypotenuse}$	trigonometric ratio	trigonometric function
$y=f(x)$	assignment	function
dy/dx	differentiation	derivative
$\int f(x)\,dx$	integration	integral
$\lim_{x\to 2} \dfrac{x^2-4}{x-2}$, $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$	tending to limit	value of limit
$\sigma \in S_n$	permuting $\{1,2,\ldots,n\}$	element of S_n
solve $(f(x)=0,x)$	solving an equation	solution of equation

Given the wide distribution of this phenomenon of symbols representing both process and concept, it is useful to provide terminology to enable it to be considered further.

COGNITIVE DEFINITION. An *elementary procept* is the amalgam of a *process*, a related con*cept* produced by that process and a symbol which represents both the process and the concept.

COGNITIVE DEFINITION. A *procept* consists of a collection of elementary procepts which have the same object (Gray & Tall, 1994).

CAVEAT. This is a *cognitive notion*, not a mathematical one. Anyone with a mathematical background might be tempted to define an elementary procept as an ordered triple (process, concept, symbol) and a procept as an equivalence class of ordered triples having the same object. Such an approach is of little cognitive value in that the purpose of the procept notion is to echo the cognitive reality of how mathematical processes are compressed mentally into manipulable mental objects. This has been the focus of attention of many researchers in mathematics education both at school and university level, including for example, Piaget (1972), Greeno (1983), Davis (1984), Dubinsky (1991), Sfard (1991), Hare & Kaput (1991). The cognitive process by which processes become conceived as manipulable objects is called *encapsulation* by Dubinsky, following Piaget.

Had the definition of *procept* been a *mathematical definition*, doubtless some mathematician would have made it before. But the procept notion implies a cognitive ambiguity — the symbol can be thought of *either* as a process, *or* as a concept. This gives a great *flexibility* in thinking — using the *process* to *do* mathematics and get answers, or using the *concept* as a compressed mental object to *think about* mathematics — in the sense of Thurston :

> I remember as a child, in fifth grade, coming to the amazing (to me) realization that the answer to 134 divided by 29 is $^{134}/_{29}$ (and so forth). What a tremendous labor-saving device ! To me, '134 divided by 29' meant a certain tedious chore, while $^{134}/_{29}$ was an object with no implicit work. I went excitedly to my father to explain my major discovery. He told me that of course this is so, a/b and a divided by b are just synonyms. To him it was just a small variation in notation.
> *(Thurston, 1990, p. 847)*

I claim that the reason why mathematicians haven't made this definition is that they *think* in such a flexible ambiguous way often without consciously realising it, but their desire for final precision is such that they write in a manner which attempts to use unambiguous definitions. This leads to the modern set-theoretic basis of mathematics in which concepts are defined as *objects*. It is a superb way to systematise mathematics but is cognitively in conflict with developmental growth in which mathematical processes *become* mathematical objects through the form of compression called encapsulation.

SEQUENTIAL AND PROCEDURAL COMPRESSION

A mathematician puts together a number of ideas in sequence to carry out a computation or a sequence of deductions in a proof using method (3). Hadamard performs such mental actions successively focusing on images before arguments are formulated logically :

> It could be supposed a priori that the links of the argument exist in full consciousness, the corresponding images being thought of by the subconscious. My personal introspection undoubtedly leads me to