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This principle will help students to become autonomous thinkers, and to become responsible for their own learning. Dubinsky & Leron use the programming language ISETL (Interactive SET Language) to get the students to engage in programming mathematical constructs in group theory and ring theory. Because the programming language is close to mathematical notation, it enables the students to construct abstract concepts like cosets and Lagrange's theorem in a concrete manner, showing considerable success in what is traditionally a difficult area.

A possible difference between this learning and the thinking of formal mathematicians is intimated by Thurston (1994, p. 167) who suggests that

> ... as new batches of mathematicians learn about the subject they tend to interpret what they read and hear more literally, so that the more easily recorded and communicated formalism and machinery tend to gradually take over from other modes of thinking.

Reflective thinking on these matters is an indispensible part of research mathematics. But it is rarely taught to undergraduates, where the focus is on content of lecture courses. At the school level problem-solving is a central part of the NCTM standards in the USA, and mathematical investigations are part of the British mathematics curriculum. Perhaps now is the time to introduce the study of mathematical thinking itself into university courses.

Of the three cognitive principles mentioned, the first essentially warns that those who have reached a greater level of maturity may have forgotten how they learnt. We therefore consider the other two principles in detail, first the nature of mathematical compression, and then move on to the process of how to teach reflective mathematical thinking.

THE COMPRESSION OF KNOWLEDGE IN MATHEMATICS

There are various methods of compression of knowledge in mathematics, including:

(1) representing information visually (a picture is worth a thousand words),

(2) using symbols to represent information compactly,

(3) if a process is too long to fit in the focus of attention, practise can make it routine so that it no longer requires much conscious thought.

Method (1) is used by many (but not all) mathematicians. In his classic study of how mathematicians do research, Hadamard explained that, with certain exceptions:

> ... mathematicians born or resident in America, whom I asked, ... practically all... — contrary to what occasional inquiries had suggested to Galton as to the man in the street — avoid not only the use of mental words, but also, just as I do, the mental use of algebraic or any other precise signs; also as in my case, they use vague images.
>
> *(Hadamard, 1945, 83-84)*

Einstein reported that visual, kinetic and other imagery proved useful in his research:

> The psychical entities which seem to serve as elements in thought are certain signs and more or less clear images which can be "voluntarily" reproduced and combined. ... The above mentioned elements are in my case, of visual and some of muscular type. Conventional words or other signs have to be sought for laboriously only in a secondary stage, when the mentioned associative play is sufficiently established and can be reproduced at will.
>
> *(Albert Einstein, in a letter to Hadamard, 1945, 142-3)*

In recent interviews with research mathematicians, Sfard (1994) found exactly the same phenomena. One mathematician reported to her:

> "To understand a new concept I must create an appropriate metaphor. A personification. Or a spatial metaphor. A metaphor of structure. Only then I can answer questions, solve problems. I may even be able then to perform some manipulations on the concept. Only when I have the metaphor. Without the metaphor I just can't do it.'
>
> 'In the structure [which he created in his mind in the attempt to understand], there are spatial elements. Many of them. It's strange, but the truth is that my student also has noticed it... a great many spatial elements. And we are dealing here with the most abstract things one can think about! Things that have nothing to do with geometry, [that are] devoid of anything physical... The way we think is always by means of something spatial... Like in 'This concept is above this one' or 'Let's move along this axis or along the other one'. There are no axes in the problem, and still...'
>
> *(Sfard, 1994)*

Mathematicians may use images in this way to relate ideas in their highly developed cognitive structure. Such *thought experiments* are highly advantageous in contemplating possible relationships before the question of logical proof arises. But it is necessary, as Hadamard said, to be *"guided by images without being enslaved by them"* (ibid, p. 88).

Students do not have such a developed cognitive structure and instead they may be deceived by their imagery. They already have their own concept images developed through previous experience (Tall & Vinner, 1981). Such imagery is often in conflict with the formal theory (see Tall, 1991*a*, 1992 for surveys). Even though concepts are given formal definitions in university mathematics, students may appeal to this imagery and infer theorems through the use of their own thought experiments. For instance, "continuous" might carry the inference of something "going on without a break", so a continuous function must clearly pass through all intermediate values, and must also be bounded and attain its bounds. For a proof by thought experiment, just imagine a picture and see.

VISUALISING MATHEMATICAL CONCEPTS

Although the private images of mathematicians may be difficult to communicate, public images, such as diagrams and graphs enable a great deal of information to be embodied in a single figure. Software which allows visual representations to be controlled by the user, to see dynamic relationships make even more powerful use of visualisation. Having been fascinated by the non-standard idea that a differentiable function infinitely magnified looks like a straight line (within infinitesimals), I wrote computer programs to look at computer drawn graphs under high magnification (figure 1). This allows a visual approach to the notion of differentiability. By using fractals such as the Takagi function (Takagi, 1903) — rechristened the "blancmange" function because of its similarity to a wobbly English milk jelly — functions could be drawn which *never* magnified to look straight (figure 2), hence intimating the notion of a *nowhere differentiable function.* Indeed, a visual proof of this argument is easy to give (Tall, 1982). By taking a small version of the blancmange function $bl(x)$, say $w(x) = bl(1000x)/1000$, for any differentiable function $f(x)$, consider the graph of $f(x) + w(x)$. This looks the same on the computer screen to a normal magnification, but under high magnification