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COMMISSION INTERNATIONALE DE L'ENSEIGNEMENT MATHÉMATIQUE

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UNDERSTANDING THE PROCESSES OF ADVANCED MATHEMATICAL THINKING [1])

An invited ICMI lecture at the *International Congress of Mathematicians* (Zurich, August 1994)

by David TALL

INTRODUCTION

In preparing successive generations of mathematicians to think in a creative mathematical way, it is difficult to convey the personal thought processes which mathematicians use themselves. So many students, unable to cope with the complexity, resort to rote-learning to pass examinations. In this paper I shall consider the growth of mathematical knowledge and the problems faced by students at university. If they are given opportunities to develop mathematical thinking processes, albeit with initially easier mathematics, they may develop attitudes to mathematics more in line with those preferred by mathematicians while standard mathematics lectures designed to "get through the material" may force them into the very kind of rote-learning that mathematicians abhor.

THE DEVELOPMENT OF MATHEMATICAL THINKING

Mathematicians struggle with ideas in research, but the ideas taught to undergraduates have been organised in a clear and logical sequence. Why is it that, when presented with these well-organised theories, students struggle too ? Is it just students' lack of effort or intellect, or are there other reasons ?

[1]) The author wishes to thank Yudariah Binte Mohammad Yusof for her research used in this presentation and Tommy Dreyfus, Eddie Gray & Anna Sfard for helpful suggestions.

AXIOM I. All mathematicians are born at age 0.

AXIOM II. To reach the age M of mathematical maturity, the mathematician must pass through ages $0, 1, 2, \ldots, M—1$.

THEOREM. A cognitive development is necessary to become a mathematician.

Proof. Since no child aged 0 has produced any important mathematical theorem, something happens between ages 0 and M that makes mathematical thinking possible. $\square$

This "proof" which caricatures a mathematical style is perhaps amusing but certainly mathematically flawed. The non-existence of a known counter-example is clearly insufficient to prove something. But if we think in *mathematical* terms about how humans think, our arguments are also liable to fail. This happened, for example, in the set-theoretic approach to school mathematics in the sixties when the apparently obvious route of introducing modern mathematics into schools failed to produce the understanding that was expected. It is therefore clear that we must take the nature of *cognitive* growth much more seriously if we are to understand the development of mathematical thinking. I propose to do this by hypothesising fundamental cognitive principles and considering the consequences.

COGNITIVE PRINCIPLE I. For survival in a Darwinian sense, the individual must maximise the use of his/her cognitive structure by focusing on concepts and methods that *work*, discarding earlier intermediate stages that no longer have value.

COROLLARY. The individual is likely to *forget* much of the learning passed through in years $0, 1, \ldots, M - 1$ and the mathematician is likely to attempt to teach current methods that work for *him*/her, not methods that will work for the student.

> One finally masters an activity so perfectly that the question of how and why students don't understand them is not asked anymore, cannot be asked anymore and is not even understood anymore as a meaningful and relevant question. *(Freudenthal, 1983, p. 469)*

> After mastering mathematical concepts, even after great effort, it becomes very hard to put oneself back into the frame of mind of someone to whom they are mysterious. *(Thurston, 1994, p. 947)*

This is not something that should cause embarassment to mathematicians, for it is sensible for a professional to do everything to climb to the summit of his or her profession. But it does suggest that there is need for professionals of a possibly different kind to devote attention to the cognitive growth of mathematical thinking to help the next generation to scale similar heights.

To understand cognitive growth it is useful to consider a second principle, which may seem initially to have little to do with mathematics, but proves in practice to have *everything* to do with its underlying power of mathematical thinking:

COGNITIVE PRINCIPLE II. The brain has a small focus of attention and a huge space for storage and therefore cognitive growth needs to develop:

(a) a mechanism for compression of ideas to fit in the focus of attention.

(b) a mechanism for linking with relevant stored information and bringing it to the focus of attention in an appropriate sequence.

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics. *(Thurston, 1990, p. 847)*

But how do we help growing mathematicians to achieve these levels of compression? Simply *telling* them the theory proves sadly to be insufficient:

> ... in their university lectures they had been given formal lectures that had not conveyed any intuitive meaning; they had passed their examinations by last-minute revision and by rote.
>
> *(W. W. Sawyer 1987, p. 61)*

To help students become mathematicians I hypothesise we need to provide them with an environment in which they can construct their own knowledge from experience and learn to *think mathematically*:

COGNITIVE PRINCIPLE III. A powerful agent in learning with understanding is by going through mathematical constructions for oneself and then *reflecting* on one's own knowledge — *thinking about thinking.*

> We believe that people learn best by *doing* and *thinking* about what they do. The abstract and the formal should be firmly based on *experience.* *(Dubinsky & Leron, 1994, p. xiv)*

This principle will help students to become autonomous thinkers, and to become responsible for their own learning. Dubinsky & Leron use the programming language ISETL (Interactive SET Language) to get the students to engage in programming mathematical constructs in group theory and ring theory. Because the programming language is close to mathematical notation, it enables the students to construct abstract concepts like cosets and Lagrange's theorem in a concrete manner, showing considerable success in what is traditionally a difficult area.

A possible difference between this learning and the thinking of formal mathematicians is intimated by Thurston (1994, p. 167) who suggests that

> ... as new batches of mathematicians learn about the subject they tend to interpret what they read and hear more literally, so that the more easily recorded and communicated formalism and machinery tend to gradually take over from other modes of thinking.

Reflective thinking on these matters is an indispensible part of research mathematics. But it is rarely taught to undergraduates, where the focus is on content of lecture courses. At the school level problem-solving is a central part of the NCTM standards in the USA, and mathematical investigations are part of the British mathematics curriculum. Perhaps now is the time to introduce the study of mathematical thinking itself into university courses.

Of the three cognitive principles mentioned, the first essentially warns that those who have reached a greater level of maturity may have forgotten how they learnt. We therefore consider the other two principles in detail, first the nature of mathematical compression, and then move on to the process of how to teach reflective mathematical thinking.

THE COMPRESSION OF KNOWLEDGE IN MATHEMATICS

There are various methods of compression of knowledge in mathematics, including:

(1) representing information visually (a picture is worth a thousand words),

(2) using symbols to represent information compactly,

(3) if a process is too long to fit in the focus of attention, practise can make it routine so that it no longer requires much conscious thought.