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(3.1) THEOREM. *The  $A[G]$ -lattices  $B$  and  $A[G]$  are factor equivalent.*

*Proof.* Define a  $B[G]$ -module structure on  $B \otimes_A B$  by letting  $B$  act on the left factor and  $G$  on the right. We will show first that  $B \otimes_A B$  and  $B[G]$  are factor equivalent as  $B[G]$ -lattices. Define the canonical  $B[G]$ -linear map  $\varphi: B \otimes_A B \rightarrow B[G]$  by

$$x \otimes y \mapsto \sum_{\sigma \in G} x\sigma(y) \cdot \sigma^{-1}.$$

Let  $H$  be a subgroup of  $G$ . If  $\sigma_1, \dots, \sigma_n$  are the  $K$ -embeddings of  $L^H$  in  $L$ , and if there is an  $A$ -basis  $\omega_1, \dots, \omega_n$  of  $B^H$ , then the restriction  $(B \otimes_A B)^H \rightarrow B[G]^H$  of  $\varphi$  is a  $B$ -linear map with matrix  $(\sigma_i(\omega_j))_{ij}$  on the bases  $\{1 \otimes \omega_j\}$  and  $\{b_i\}$ , where  $b_i$  is the formal sum of those  $\sigma \in G$  for which  $\sigma^{-1}$  restricts to  $\sigma_i$ . The square of the determinant of this matrix generates the discriminant  $\Delta(B^H/A)$  as an  $A$ -ideal. By localization it follows that even if  $B$  is not free over  $A$ , we have

$$[B[G]^H : \varphi(B \otimes_A B)^H]_B^2 = \Delta(B^H/A) \cdot B.$$

By Hasse's conductor discriminant product formula [15, Ch. VI, §3] the ideal  $\Delta(B^H/A)$  is a factorizable function of  $H$ , so  $B \otimes_A B$  and  $B[G]$  are factor equivalent  $B[G]$ -lattices.

In order to descend to  $A[G]$ -lattices, note that there exists an  $A[G]$ -linear injection  $i: A[G] \rightarrow B$  by the normal basis theorem, and consider the induced  $B[G]$ -linear map  $i_*: B[G] \rightarrow B \otimes_A B$  that sends  $b\sigma$  to  $b \otimes i(\sigma)$  for  $b \in B$  and  $\sigma \in G$ . We have

$$[(B \otimes_A B)^H : i_*(B[G])^H]_B = [B^H : i(A[G])^H]_A \cdot B,$$

and by (2.5) we know that the left hand side is a factorizable function of  $H$ . But then the  $A$ -index  $[B^H : i(A[G])^H]_A$  is also factorizable.  $\square$

#### 4. S-UNITS

Let  $L/K$  be a Galois extension of number fields with Galois group  $G$ , and let  $S$  be a finite  $G$ -stable set of primes of  $L$  containing the infinite primes. The ring of  $S$ -integers of  $L$  consists of all elements of  $L$  that are integral outside  $S$ . Its class number is written as  $h_S(L)$  and its unit group, the group of  $S$ -units of  $L$ , is denoted by  $U_S(L)$ . The group of roots of unity in  $L$  is denoted by  $\mu_L$  and its order is written as  $w(L)$ .

Define the  $\mathbf{Z}[G]$ -lattice  $X_S$  to be the kernel of the map  $\mathbf{Z}[S] \rightarrow \mathbf{Z}$  that sends each  $\mathfrak{p} \in S$  to 1. We have a canonical map  $\log_L: U_S(L) \rightarrow \mathbf{R} \otimes_{\mathbf{Z}} X_S$  sending  $x$  to the formal sum  $\sum_{\mathfrak{p} \in S} (\log |x|_{\mathfrak{p}}) \otimes \mathfrak{p}$  in  $\mathbf{R}[S]$ . Here the normalization of the valuation at a prime  $\mathfrak{p}$  of  $L$ , lying over a prime  $p$  of  $\mathbf{Q}$ , is given by  $|u|_{\mathfrak{p}} = |N_{L_{\mathfrak{p}}/\mathbf{Q}_p}(u)|_p$ , where  $|\cdot|_p$  is the usual valuation on the completion  $\mathbf{Q}_p$  of  $\mathbf{Q}$  (with  $\mathbf{Q}_p = \mathbf{R}$  if  $p = \infty$ ). Dirichlet's unit theorem says that  $\log_L$  embeds  $U_S(L)/\mu_L$  as a discrete cocompact lattice in  $\mathbf{R} \otimes_{\mathbf{Z}} X_S$ . The  $S$ -regulator  $R_S(L) \in \mathbf{R}_{>0}$  is defined to be the covolume of this lattice when the measure on  $\mathbf{R} \otimes_{\mathbf{Z}} X_S$  is normalized to give  $1 \otimes X_S$  covolume 1.

For a subgroup  $H$  of  $G$  we let  $S(H)$  be the set of primes of  $L^H$  that extend to a prime in  $S$ . We will write  $h_S(L^H)$  for  $h_{S(H)}(L^H)$  and  $R_S(L^H)$  for  $R_{S(H)}(L^H)$ . Brauer [1] has shown that the function  $H \mapsto h_S(L^H)R_S(L^H)/w(L^H)$  is a factorizable function with values in  $\mathbf{R}_{>0}$ . The easiest way to see this is by noting that this quotient is the absolute value of the leading coefficient in the Taylor series expansion at  $s = 0$  of the zeta-function  $\zeta_{L^H, S}(s)$  of  $L^H$ ; see Tate [17, Ch. I, 2.2]. Since  $\zeta_{L^H, S}(s)$  is equal to the Artin  $L$ -series  $L_S(1_H^G, s)$ , the factorizability result then follows from the fact that  $L_S(\chi_1 + \chi_2, s) = L_S(\chi_1, s)L_S(\chi_2, s)$ .

The group  $G$  acts on  $S$ , so it acts on  $\mathbf{Z}[S]$  and on  $X_S$ . The map  $\log_L$  induces an  $\mathbf{R}[G]$ -linear isomorphism  $\mathbf{R} \otimes_{\mathbf{Z}} U_S(L) \xrightarrow{\sim} \mathbf{R} \otimes_{\mathbf{Z}} X_S$ . It follows that the  $\mathbf{Q}[G]$ -modules  $\mathbf{Q} \otimes_{\mathbf{Z}} U_S(L)$  and  $\mathbf{Q} \otimes_{\mathbf{Z}} X_S$  are isomorphic; see [3, p. 110]. In particular, there exists a  $\mathbf{Z}[G]$ -linear embedding  $i: X_S \rightarrow U_S(L)$ .

For a prime  $\mathfrak{p}$  of  $L^H$  all primes  $\mathfrak{q}$  of  $L$  lying over  $\mathfrak{p}$  have the same local degree, which we denote by  $n_{\mathfrak{p}}(L/L^H)$ . Let  $n(H)$  be the product of all  $n_{\mathfrak{p}}(L/L^H)$  with  $\mathfrak{p} \in S(H)$ , and let  $l(H)$  be their least common multiple.

(4.1) THEOREM. *For any  $\mathbf{Z}[G]$ -linear embedding  $i: X_S \hookrightarrow U_S(L)$ , the function*

$$H \mapsto [U_S(L)^H : i(X_S)^H] \frac{n(H)}{l(H) h_S(L^H)}$$

*with values in  $\mathbf{Q}_{>0}$  is factorizable.*

*Proof.* For  $\mathbf{Z}$ -lattices  $L_1, L_2$  spanning the same real vector space  $V$  we define the "index"  $[L_2 : L_1] \in \mathbf{R}_{>0}$  as follows: choose a Haar measure on  $V$  such that  $L_2$  has covolume 1 and let  $[L_2 : L_1]$  be the covolume of  $L_1$ . Note that this notion coincides with the usual index in the case that  $L_1 \subset L_2$ , and that  $[L_1 : L_2][L_2 : L_3] = [L_1 : L_3]$ . Moreover, for any  $\mathbf{R}$ -linear automorphism  $\varphi$  of  $V$  we have  $[L_1 : \varphi(L_1)] = |\det \varphi|$ .

For each subgroup  $H$  of  $G$  we have an injective map  $j_H: \mathbf{Z}[S(H)] \rightarrow \mathbf{Z}[S]$  sending  $\mathfrak{p}$  to  $\sum_{\mathfrak{q}|\mathfrak{p}} n_{\mathfrak{p}}(L/L^H) \cdot \mathfrak{q}$ . This map respects the logarithm map in the sense that we have a commutative diagram

$$\begin{array}{ccc} U_S(L^H) & \xrightarrow{\log_{L^H}} & \mathbf{R} \otimes X_{S(H)} \\ \parallel & & \downarrow 1 \otimes j_H \\ U_S(L)^H & \xrightarrow{\log_L} & \mathbf{R} \otimes X_S^H, \end{array}$$

where the vertical map on the left is inclusion. We therefore have

$$R_S(L^H) = [X_{S(H)} : \log_{L^H} U_S(L^H)] = \frac{[X_S^H : \log_L U_S(L)^H]}{[X_S^H : j_H(X_{S(H)})]}.$$

The composite map  $X_S \xrightarrow{i} U_S(L) \xrightarrow{\log_L} \mathbf{R} \otimes X_S$  induces an  $\mathbf{R}[G]$ -linear automorphism  $\varphi$  of  $\mathbf{R} \otimes X_S$ . With the notation of (2.3) one has

$$|d_{\varphi}(H)| = [X_S^H : \varphi(X_S^H)] = [X_S^H : \log_L U_S(L)^H] \frac{[U_S(L)^H : i(X_S^H)]}{w(L^H)}.$$

Combining these two formulas, and dividing by  $h_S(L^H)$ , we get

$$(4.2) \quad [U_S(L)^H : i(X_S^H)] \frac{[X_S^H : j_H(X_{S(H)})]}{h_S(L^H)} = |d_{\varphi}(H)| \frac{w(L^H)}{h_S(L^H) R_S(L^H)}.$$

The right hand side is factorizable by (2.3) and Brauer's theorem. It remains to show that  $[X_S^H : j_H(X_{S(H)})] = n(H)/l(H)$ . In order to do this we compare the sequence defining  $X_{S(H)}$  with the  $H$ -invariants of the sequence defining  $X_S$ :

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_{S(H)} & \longrightarrow & \mathbf{Z}[S(H)] & \longrightarrow & \mathbf{Z} \longrightarrow 0 \\ & & \downarrow & & \downarrow j_H & & \downarrow \#H \\ 0 & \longrightarrow & X_S^H & \longrightarrow & \mathbf{Z}[S]^H & \longrightarrow & \mathbf{Z}. \end{array}$$

The rows in this commutative diagram are exact and the vertical maps are injective. The cokernel  $C$  of the map  $j_H$  is the group  $\bigoplus_{\mathfrak{p} \in S(H)} \mathbf{Z}/n_{\mathfrak{p}}(L/L^H)\mathbf{Z}$ , which has order  $n(H)$ . It is not hard to see that the image of  $C$  in the cokernel of the rightmost vertical map has order  $l(H)$ . With the snake lemma our claim follows.  $\square$

(4.3) REMARK. One can shorten this proof somewhat by using results in Tate's book on the Stark conjectures. Tate shows in [17, Ch. II, 1.1] that  $[U_S(L) : i(X_S)]/h_S(L)$  is equal to the Stark-quotient  $A(1, i)$ , where 1 denotes the trivial character of the trivial Galois group of  $L$  over  $L$ . Compatibility of the Stark-quotient with respect to inflation and addition of characters implies that the number on the left in (4.2) equals  $A(1_H^G, i)$ , and that it is a factorizable function of  $H$ .

(4.4) REMARK. In order to say that (4.1) determines the factor equivalence class of  $U_S(L)$  we should define factor equivalence for  $\mathbf{Z}[G]$ -modules with  $\mathbf{Z}$ -torsion. This can be done by replacing condition (2) in (2.4) by the condition that the quotient of the order of cokernel and kernel of the map  $M^H \rightarrow N^H$  should be factorizable.

Alternatively, one can look at  $\bar{U}(L) = U_S(L)/\mu_L$  instead of  $U_S(L)$ . This approach does introduce new factors into the formula because  $\bar{U}(L)^H$  is not necessarily equal to  $\bar{U}(L^H)$ . More precisely,  $c(H) = [\bar{U}(L)^H : \bar{U}(L^H)]$  is the order of the kernel of the map  $H^1(H, \mu_L) \rightarrow H^1(H, U_S(L))$ , so we know that it is built up from primes dividing both  $w(L)$  and  $\#G$ . For  $\mathbf{Z}[G]$ -embeddings  $i: X_S \hookrightarrow \bar{U}(L)$  it turns out that the map

$$(4.5) \quad H \mapsto [\bar{U}(L)^H : i(X_S)^H] \frac{w(L^H) n(H)}{h_S(L^H) c(H) l(H)}$$

is factorizable. Thus one recovers [4, §3, Th. 3], where it is assumed that  $L$  has odd degree over  $K$  and  $K$  is totally real, so that  $c(H) = n(H) = l(H) = 1$  and  $w(L^H) = 2$ . Brauer [1] showed that the odd part of  $w(L^H)$  is a factorizable  $\mathbf{Q}^*$ -valued function of  $H$ , and his argument inspired the following lemma (cf. [11, Prop. 4.7]).

(4.6) LEMMA. *Let  $G$  be a group, let  $D$  be a subgroup of  $G$  and let  $N$  be a normal subgroup of  $D$  of index  $n$  such that  $D/N$  is cyclic. For every divisor  $d$  of  $n$  and subgroup  $H$  of  $G$ , let  $m_d(H) \in \mathbf{Z}$  be the number of  $D$ -orbits of  $G/H$  that split up into exactly  $d$  orbits under the action of  $N$ . Then  $m_d(H)$  is a factorizable  $\mathbf{Z}$ -valued function of  $H$ .*

*Proof.* Let  $\chi: D \rightarrow \mathbf{C}^*$  be a complex linear character such that  $\chi(N) = 1$ , and let  $\chi^G$  be the induced character of  $G$ . We claim that  $\langle \chi^G, 1_H^G \rangle_G$  is the sum of those  $m_d(H)$  for which  $d$  is a multiple of the order of  $\chi$ . Since  $\langle \cdot, \cdot \rangle_G$  is a bilinear operation on characters of  $G$  (see [16, §7.2]) the integer  $\langle \chi^G, 1_H^G \rangle_G$  is a factorizable function of  $H$ . We deduce the lemma from the claim by

taking  $\chi$  of order  $d$  and using induction: we start with  $n = d$  and then successively remove prime factors from  $d$ . It remains to show the claim.

By Frobenius reciprocity one has  $\langle \chi^G, 1_H^G \rangle_G = \langle \chi, 1_H^G|_D \rangle_D$ , which is equal to the multiplicity of  $\chi$  in the complex representation  $\mathbf{C}[G/H]$  of  $D$ . The  $D$ -set  $G/H$  is  $D$ -isomorphic to a disjoint union  $\coprod_X D/D_X$ , where  $X$  runs over the  $D$ -orbits of  $G/H$ , and each  $D_X$  is a subgroup of  $D$ . The multiplicity of  $\chi$  in  $\mathbf{C}[D/D_X]$  is either 0 or 1, and it is 1 if and only if  $D_X \subset \text{Ker } \chi$ . Since  $N \subset \text{Ker } \chi$ , and  $D/N$  is cyclic, it follows that  $\langle \chi^G, 1_H^G \rangle_G$  is equal to the number of  $X$  for which the order of  $\chi$  divides  $[D : ND_X]$ . This index is the number of  $N$ -orbits of  $D/D_X$ , so the claim follows.  $\square$

If for a prime number  $p$  the roots of unity in  $L$  of  $p$ -power order generate a cyclic extension of  $K$ , then one can show with the lemma (with  $D = G$ ) that the  $p$ -part of  $w(L^H)$  is a factorizable  $\mathbf{Q}^*$ -valued function of  $H$ . The condition holds for all  $p > 2$ , so the odd part of  $w(L^H)$  is factorizable.

For any prime  $\mathfrak{p}$  of  $K$  and  $d \in \mathbf{Z}$  the number of primes in  $L^H$  extending  $\mathfrak{p}$  with residue degree  $d$  is a  $\mathbf{Z}$ -valued factorizable function of  $H$ . This follows from the lemma if we take  $D$  and  $N$  to be the decomposition group and the inertia group of  $\mathfrak{p}$ . If  $\mathfrak{p}$  has a cyclic decomposition group  $D$  then one can also take  $N = 1$ , and deduce the same statement with “residue degree” replaced by “local degree”.

It follows that the factor  $n(H)$  in (4.1) can be replaced by the product of the ramification indices in the extension  $L/L^H$  of those primes  $\mathfrak{p} \in S(H)$  that extend to a prime of  $L$  with non-cyclic decomposition group in  $L/K$ . In particular,  $n(H)$  is factorizable if  $S$  contains no finite ramified primes.

## 5. APPLICATIONS

Without giving proofs we indicate some concrete applications of the factor equivalence results given in the last two sections.

(5.1) CYCLIC SUBFIELD INTEGER INDEX. Let  $K$  be a Galois extension of  $\mathbf{Q}$  with abelian Galois group  $G$  and ring of integers  $\mathcal{O}_K$ . For a  $\mathbf{Z}[G]$ -module  $M$  let  $c_G(M)$  be the index in  $M$  of  $\sum M^H$ , where the sum is taken over those subgroups  $H$  of  $G$  for which  $G/H$  is cyclic. In particular,