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semidirect product of K and a connected simply-connected nilpotent Lie group and Γ is a discrete cocompact subgroup of G [12, Theorem 6.4]. We may as well assume that $X = \Gamma \backslash G/K$. By passing to a finite cover, we may assume that K is trivial. That is, X is a nilmanifold. From [27, Corollary 7.28], $H^p(X; \mathbf{C}) \cong H^p(\mathfrak{g}, \mathbf{C})$, the Lie algebra cohomology of \mathfrak{g} . From [7], $H^p(\mathfrak{g}, \mathbf{C}) \neq 0$ for all $p \in [0, \dim(X)]$. Thus for all $p \in [0, \dim(X)]$, $H^p(X; \mathbf{C}) \neq 0$.

Now let ω be a nonzero harmonic p -form on X . Let $\pi^*\omega$ be its pullback to \tilde{X} . The idea is to construct low-energy square-integrable p -forms on X by multiplying $\pi^*\omega$ by appropriate functions on X . We define the functions as in [2, §2]. Take a smooth triangulation of X and choose a fundamental domain F for the lifted triangulation of \tilde{X} . If E is a finite subset of $\pi_1(X)$, let χ_H be the characteristic function of $H = \bigcup_{g \in E} g \cdot F$. Given numbers $0 < \epsilon_1 < \epsilon_2 < 1$, choose a nonincreasing function $\psi \in C_0^\infty([0, \infty))$ which is identically one on $[0, \epsilon_1]$ and identically zero on $[\epsilon_2, \infty)$. Define a compactly-supported function f_E on \tilde{X} by $f_E(m) = \psi(d(m, H))$. Then there is a constant $C_1 > 0$, independent of E , such that

$$(5.12) \quad \int_{\tilde{X}} |df_E|^2 \leq C_1 \text{area}(\partial H).$$

Define $\rho_E \in \Lambda^p(\tilde{X})$ by $\rho_E = f_E \cdot \pi^*\omega$. We have $d\rho_E = df_E \wedge \pi^*\omega$ and $d^*\rho_E = -i(df_E)\pi^*\omega$. As f_E is identically one on H , it follows that there is a constant $C > 0$, independent of E , such that

$$(5.13) \quad \frac{\int_{\tilde{X}} [|d\rho_E|^2 + |d^*\rho_E|^2]}{\int_{\tilde{X}} |\rho_E|^2} \leq C \frac{\text{area}(\partial H)}{\text{vol}(H)}.$$

As $\pi_1(X)$ is amenable, by an appropriate choice of E this ratio can be made arbitrarily small. Thus $0 \in \sigma(\Delta_p)$. \square

QUESTION. Does the conclusion of Proposition 20 hold if we only assume that $\pi_1(X)$ is amenable?

6. TOPOLOGICALLY TAME MANIFOLDS

Another class of manifolds for which one can hope to get some nontrivial results about the zero-in-the-spectrum question is given by *topologically tame* manifolds, meaning manifolds M which are diffeomorphic to the interior of a compact manifold N with boundary. If M has finite volume then $\text{Ker}(\Delta_0) \neq 0$,

so we restrict our attention to the infinite volume case. A limited result is given by Corollary 2.

An interesting class of topologically tame manifolds consists of those which are radially symmetric. This means that M is diffeomorphic to \mathbf{R}^n , with a metric which is given on $\mathbf{R}^n - \{0\} \cong (0, \infty) \times S^{n-1}$ by

$$(6.1) \quad g = dr^2 + \phi^2(r)d\Omega^2.$$

Here $d\Omega^2$ is the standard metric on S^{n-1} , $r \in (0, \infty)$, $\phi \in C^\infty([0, \infty))$, $\phi(0) = 0$, $\phi'(0) = 1$ and $\phi(r) > 0$ for $r > 0$.

PROPOSITION 21. *Suppose that there is a constant $c \geq 0$ such that $\text{Ricci}_M \geq -c^2$. Then $0 \in \sigma(\Delta_p)$ for some p .*

Proof. We may assume that $\text{vol}(M) = \infty$. Suppose first that

$$\liminf_{r \rightarrow \infty} \phi(r) < \infty.$$

Then there is a constant $C > 0$ and a sequence $\{r_j\}_{j=1}^\infty$ such that $\lim_{j \rightarrow \infty} r_j = \infty$ and $\phi(r_j) \leq C$. Let D_j be the domain in M defined by $r \leq r_j$. Then $\text{area}(D_j) \leq C^{n-1} \text{vol}(S^{n-1})$ and $\lim_{j \rightarrow \infty} \text{vol}(D_j) = \infty$. Thus M is not open at infinity. By Proposition 6, $0 \in \sigma(\Delta_0)$.

Now suppose that $\liminf_{r \rightarrow \infty} \phi(r) = \infty$. We want to show that M is hyperEuclidean and apply Proposition 7. Consider a map $F : M \rightarrow \mathbf{R}^n$ given in polar coordinates by

$$(6.2) \quad F(r, \theta) = (s(r), \theta),$$

for some $s : [0, \infty) \rightarrow [0, \infty)$. The condition for F to be distance-nonincreasing is

$$(6.3) \quad |s'(r)| \leq 1, \quad s(r) \leq \phi(r).$$

If $\lim_{r \rightarrow \infty} s(r) = \infty$ then F is a proper map of degree one. It remains to construct s satisfying (6.3).

Put

$$(6.4) \quad \tilde{\phi}(r) = \inf_{v \in [r, \infty)} \phi(v).$$

Replacing ϕ by $\tilde{\phi}$, we may assume that ϕ is monotonically nondecreasing. Thinking of $\phi(r)$ as representing the trajectory of a car in front of us which is blocking the road, with our car's velocity bounded above by one, it is intuitively clear that we can find a trajectory $s(r)$ for our car such that

$\lim_{r \rightarrow \infty} s(r) = \infty$. More precisely, let $\rho \in C^\infty([0, 2])$ be a nondecreasing function which is identically zero near 0, identically one near 2 and satisfies $\rho'(x) \leq 1$ for all $x \in [0, 2]$. Put $r_0 = 0$ and define $\{r'_j\}_{j=0}^\infty$ and $\{r_j\}_{j=1}^\infty$ inductively by

$$(6.5) \quad \begin{aligned} r'_j &= \inf\{r : r \geq r_j + 2 \text{ and } \phi(r) \geq j + 1\}, \\ r_{j+1} &= r'_j + 2. \end{aligned}$$

Define s by

$$(6.6) \quad s(r) = \begin{cases} j & \text{if } r \in [r_j, r'_j] \\ j + \rho(r - r'_j) & \text{if } r \in [r'_j, r_{j+1}]. \end{cases}$$

Then s satisfies (6.3) and $\lim_{r \rightarrow \infty} s(r) = \infty$. \square

QUESTION. What can one say in the radially symmetric case without the assumption of a lower bound on the Ricci curvature?

Another interesting class of topologically tame manifolds consists of those which are hyperbolic, that is, of constant sectional curvature -1 . Complete hyperbolic manifolds are divided into those which are *geometrically finite* and those which are *geometrically infinite*. Roughly speaking, M is geometrically finite if its set of ends consists of a finite number of standard cusps and flares.

PROPOSITION 22 (Mazzeo-Phillips [23, Theorem 1.11]). *Let M be an infinite-volume geometrically finite hyperbolic manifold. If $\dim(M) = 2k$ then $\dim(\text{Ker}(\Delta_k)) = \infty$. If $\dim(M) = 2k + 1$ then $\sigma(\Delta_k) = \sigma(\Delta_{k+1}) = [0, \infty)$.*

The paper [23] also computes $\dim(\text{Ker}(\Delta_p))$ for such manifolds.

In general, geometrically infinite hyperbolic manifolds can have wild end behavior. However, in three dimensions one can show that the ends have a fairly nice structure. This is used to prove the next result.

PROPOSITION 23 (Canary [4, Theorem A]). *If M is a geometrically infinite topologically tame hyperbolic 3-manifold then $0 \in \sigma(\Delta_0)$.*

Proof. The method of proof is to show that M is not open at infinity and then apply Theorem 6. See [4] for details. \square

Thus zero lies in the spectrum of all topologically tame hyperbolic 3-manifolds. From Proposition 2, the same statement is true for compactly-supported modifications of such manifolds.

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