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a. $b_0^{(2)} \neq 0 : S^4,\ S^2 \times S^2,\ \mathbf{C}P^2$.

b. $0 \in \sigma\big(\triangle_0 \text{ on } \Lambda^0/\operatorname{Ker}(d)\big) : \mathbf{R}^4,\ S^3 \times \mathbf{R},\ S^2 \times \mathbf{R}^2,\ Nil^3 \times \mathbf{R},\ Nil^4,\ Sol_0^4,$ $Sol_1^4,\ Sol_{m,n}^4$.

c. $b_1^{(2)} \neq 0 : S^2 \times H^2$.

d. $0 \in \sigma\big(\triangle_1 \text{ on } \Lambda^1/\operatorname{Ker}(d)\big) : H^3 \times \mathbf{R},\ \widetilde{SL_2} \times \mathbf{R},\ H^2 \times \mathbf{R}^2$.

e. $\chi > 0 : H^4,\ H^2 \times H^2,\ \mathbf{C}H^2$.

Part 2. of the proposition follows.

3. Suppose that zero is not in the spectrum of $\widetilde{X}$. From Properties 7 and 9, $\chi(X) = \tau(X) = 0$. From the classification of complex surfaces, X has a geometric structure [32, p. 148–149]. This contradicts part 2. of the proposition. $\square$

5.4 More Dimensions

In this subsection we give some partial positive results about the zero-in-the-spectrum question for covers of compact manifolds of arbitrary dimension. Let us first recall some facts about index theory [18]. Let X be a closed Riemannian manifold. If $\dim(X)$ is even, consider the operator $d + d^*$ on $\Lambda^*(X)$. Give $\Lambda^*(X)$ the $\mathbf{Z}_2$-grading coming from (3.12). Then the signature $\tau(X)$ equals the index of $d+d^*$. To say this in a more complicated way, the operator $d+d^*$ defines a element $[d+d^*]$ of the K-homology group $K_0(X)$. Let $\eta : X \to \text{pt.}$ be the (only) map from X to a point. Then $\eta_*([d+d^*]) \in K_0(\text{pt.})$. There is a map $A : K_0(\text{pt.}) \to K_0(\mathbf{C})$ which is the identity, as both sides are $\mathbf{Z}$. So we can say that $\tau(X) = A\big(\eta\big([d + d^*]\big)\big) \in K_0(\mathbf{C})$.

We now extend the preceding remarks to the case of a group action. Let M be a normal cover of X with covering group Γ. The fiber bundle $M \to X$ is classified by a map $\nu : X \to B\Gamma$, defined up to homotopy. Let $\widetilde{d}$ be exterior differentiation on M. Consider the operator $\widetilde{d} + \widetilde{d}^*$. Taking into account the action of Γ on M, one can define a refined index $\operatorname{ind}(\widetilde{d} + \widetilde{d}^*) \in K_0(C_r^*\Gamma)$, where $C_r^*\Gamma$ is the reduced group C^*-algebra of Γ.

We recall the statement of the Strong Novikov Conjecture (SNC) [18, 29]. This is a conjecture about a countable discrete group Γ, namely that the assembly map $A : K_*(B\Gamma) \to K_*(C_r^*\Gamma)$ is rationally injective. Many groups of a geometric origin, such as discrete subgroups of connected Lie groups or Gromov-hyperbolic groups, are known to satisfy SNC. There are no known groups which do not satisfy SNC.

PROPOSITION 19. *Let X be a closed Riemannian manifold with a surjective homomorphism $\pi_1(X) \to \Gamma$. Let M be the induced normal Γ-cover of X. Suppose that Γ satisfies SNC. Let $L(X) \in \mathrm{H}^*(X;\mathbf{C})$ be the Hirzebruch L-class of X and let $*L(X) \in \mathrm{H}_*(X;\mathbf{C})$ be its Poincaré dual. Then if $\nu_*(*L(X)) \neq 0$ in $H_*(B\Gamma;\mathbf{C})$, zero lies in the spectrum of M. In fact, $0 \in \sigma\left(\triangle_{\frac{dim(X)}{2}}\right)$ if* $\dim(X)$ *is even and $0 \in \sigma\left(\triangle_{\frac{dim(X)\pm 1}{2}}\right)$ if* $\dim(X)$ *is odd.*

Proof. Suppose first that $\dim(X)$ is even. Suppose that zero does not lie in the spectrum of M. Then the operator $\widetilde{d}+\widetilde{d}^*$ is invertible. (More precisely, it is invertible as an operator on a Hilbert $C_r^*\Gamma$-module of differential forms on M.) This implies that $\mathrm{ind}(\widetilde{d}+\widetilde{d}^*)$ vanishes in $K_0(C_r^*\Gamma)$.

The higher index theorem says that

$$\mathrm{ind}(\widetilde{d}+\widetilde{d}^*) = A\big(\nu_*\left([d+d^*]\right)\big) . \tag{5.10}$$

Let $A_{\mathbf{C}} : K_0(B\Gamma)\otimes\mathbf{C} \to K_0(C_r^*\Gamma)\otimes\mathbf{C}$ be the complexified assembly map. Using the isomorphism $K_0(B\Gamma)\otimes\mathbf{C} \cong \mathrm{H}_{even}(B\Gamma;\mathbf{C})$, the higher index theorem implies that in $K_0(C_r^*\Gamma)\otimes\mathbf{C}$,

$$\mathrm{ind}(\widetilde{d}+\widetilde{d}^*)_{\mathbf{C}} = A_{\mathbf{C}}\big(\nu_*\left(*L(X)\right)\big) . \tag{5.11}$$

By assumption, $A_{\mathbf{C}}$ is injective. This gives a contradiction.

Let T be the operator obtained by restricting $\widetilde{d}+\widetilde{d}^*$ to

$$\Lambda^{\frac{dim(X)}{2}}(M) \oplus \overline{\widetilde{d}\Lambda^{\frac{dim(X)}{2}}(M)} \oplus \overline{*\widetilde{d}\Lambda^{\frac{dim(X)}{2}}(M)}.$$

One can show that the other differential forms on M cancel out when computing the rational index of $\widetilde{d}+\widetilde{d}^*$, so T will have the same index as $\widetilde{d}+\widetilde{d}^*$. Then the same arguments apply to T to give $0 \in \sigma\left(\triangle_{\frac{dim(X)}{2}}\right)$.

If $\dim(X)$ is odd, consider the even-dimensional manifold $X' = X \times S^1$ and the group $\Gamma' = \Gamma \times \mathbf{Z}$. As the proposition holds for X', it must also hold for X. $\square$

COROLLARY 4. *Let X be a closed Riemannian manifold. Let $[X] \in \mathrm{H}_{dim(X)}(X;\mathbf{C})$ be its fundamental class. Suppose that there is a surjective homomorphism $\pi_1(X) \to \Gamma$ such that Γ satisfies SNC and the composite map $X \to B\pi_1(X) \to B\Gamma$ sends $[X]$ to a nonzero element of $\mathrm{H}_{dim(X)}(B\Gamma;\mathbf{C})$. Let M be the induced normal Γ-cover of X. Then on M, $0 \in \sigma\left(\triangle_{\frac{dim(X)}{2}}\right)$ if* $\dim(X)$ *is even and $0 \in \sigma\left(\triangle_{\frac{dim(X)\pm 1}{2}}\right)$ if* $\dim(X)$ *is odd.*

Proof. As the Hirzebruch L-class starts out as $L(X) = 1 + \dots$, its Poincaré dual is of the form $*L(X) = \dots + [X]$. The corollary follows from Proposition 19. $\square$

COROLLARY 5. *Let X be a closed aspherical Riemannian manifold whose fundamental group satisfies SNC. Then on $\widetilde{X}$, $0 \in \sigma\left(\triangle_{\frac{\dim(X)}{2}}\right)$ if $\dim(X)$ is even and $0 \in \sigma\left(\triangle_{\frac{\dim(X)\pm 1}{2}}\right)$ if $\dim(X)$ is odd.*

Proof. This follows from Corollary 4. $\square$

EXAMPLES.

1. If $X = T^n$ then Corollary 5 is consistent with Example 2 of Section 2.
2. If X is a compact quotient of H^{2n} then Corollary 5 is consistent with Example 3 of Section 2.
3. If X is a compact quotient of H^{2n+1} then Corollary 5 is consistent with Example 4 of Section 2.
4. If X is a closed nonpositively-curved locally symmetric space then Corollary 5 is consistent with the second remark after Proposition 7.

If X is a closed aspherical manifold, it is known that SNC implies that the rational Pontryagin classes of X are homotopy-invariants [18] and that X does not admit a Riemannian metric of positive scalar curvature [29]. Thus we see that these three questions about aspherical manifolds, namely homotopy-invariance of rational Pontryagin classes, (non)existence of positive-scalar-curvature metrics and the zero-in-the-spectrum question, are roughly all on the same footing.

If X is a closed aspherical Riemannian manifold, one can ask for which p one has $0 \in \sigma(\triangle_p)$ on $\widetilde{X}$. The case of locally symmetric spaces is covered by the second remark after Proposition 7. Another interesting class of aspherical manifolds consists of those with amenable fundamental group. By [5], $\mathrm{Ker}(\triangle_p) = 0$ for all p. By Corollary 3, $0 \in \sigma(\triangle_0)$.

PROPOSITION 20. *If X is a closed aspherical manifold such that $\pi_1(X)$ has a nilpotent subgroup of finite index then $0 \in \sigma(\triangle_p)$ on $\widetilde{X}$ for all $p \in [0, \dim(X)]$.*

Proof. First, X is homotopy-equivalent to an infranilmanifold, that is, a quotient of the form $\Gamma \backslash G/K$ where K is a finite group, G is the semidirect product of K and a connected simply-connected nilpotent Lie group and Γ is a discrete cocompact subgroup of G [12, Theorem 6.4]. We may as well assume that $X = \Gamma\backslash G/K$. By passing to a finite cover, we may assume that K is trivial. That is, X is a nilmanifold. From [27, Corollary 7.28], $\mathrm{H}^p(X;\mathbf{C}) \cong \mathrm{H}^p(g,\mathbf{C})$, the Lie algebra cohomology of g. From [7], $\mathrm{H}^p(g,\mathbf{C}) \neq 0$ for all $p \in [0,\dim(X)]$. Thus for all $p \in [0,\dim(X)]$, $\mathrm{H}^p(X;\mathbf{C}) \neq 0$.

Now let ω be a nonzero harmonic p-form on X. Let $\pi^*\omega$ be its pullback to $\widetilde{X}$. The idea is to construct low-energy square-integrable p-forms on X by multiplying $\pi^*\omega$ by appropriate functions on X. We define the functions as in [2, §2]. Take a smooth triangulation of X and choose a fundamental domain F for the lifted triangulation of $\widetilde{X}$. If E is a finite subset of $\pi_1(X)$, let χ_H be the characteristic function of $H = \bigcup_{g\in E} g\cdot F$. Given numbers $0 < \epsilon_1 < \epsilon_2 < 1$, choose a nonincreasing function $\psi \in C_0^\infty\big([0,\infty)\big)$ which is identically one on $[0,\epsilon_1]$ and identically zero on $[\epsilon_2,\infty)$. Define a compactly-supported function f_E on $\widetilde{X}$ by $f_E(m) = \psi\big(d(m,H)\big)$. Then there is a constant $C_1 > 0$, independent of E, such that

$$\int_{\widetilde{X}} |df_E|^2 \leq C_1 \,\mathrm{area}(\partial H). \tag{5.12}$$

Define $\rho_E \in \Lambda^p(\widetilde{X})$ by $\rho_E = f_E \cdot \pi^*\omega$. We have $d\rho_E = df_E \wedge \pi^*\omega$ and $d^*\rho_E = -i(df_E)\pi^*\omega$. As f_E is identically one on H, it follows that there is a constant $C > 0$, independent of E, such that

$$\frac{\int_{\widetilde{X}}\left[|d\rho_E|^2 + |d^*\rho_E|^2\right]}{\int_{\widetilde{X}}|\rho_E|^2} \leq C\frac{\mathrm{area}(\partial H)}{\mathrm{vol}(H)}. \tag{5.13}$$

As $\pi_1(X)$ is amenable, by an appropriate choice of E this ratio can be made arbitrarily small. Thus $0 \in \sigma(\triangle_p)$. □

QUESTION. Does the conclusion of Proposition 20 hold if we only assume that $\pi_1(X)$ is amenable?

6. Topologically Tame Manifolds

Another class of manifolds for which one can hope to get some nontrivial results about the zero-in-the-spectrum question is given by *topologically tame* manifolds, meaning manifolds M which are diffeomorphic to the interior of a compact manifold N with boundary. If M has finite volume then $\mathrm{Ker}(\triangle_0) \neq 0$,