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- a. $b_0^{(2)} \neq 0 : S^4, S^2 \times S^2, \mathbf{CP}^2$.
- b. $0 \in \sigma(\Delta_0 \text{ on } \Lambda^0 / \text{Ker}(d)) : \mathbf{R}^4, S^3 \times \mathbf{R}, S^2 \times \mathbf{R}^2, Nil^3 \times \mathbf{R}, Nil^4, Sol_0^4, Sol_1^4, Sol_{m,n}^4$.
- c. $b_1^{(2)} \neq 0 : S^2 \times H^2$.
- d. $0 \in \sigma(\Delta_1 \text{ on } \Lambda^1 / \text{Ker}(d)) : H^3 \times \mathbf{R}, \widetilde{SL}_2 \times \mathbf{R}, H^2 \times \mathbf{R}^2$.
- e. $\chi > 0 : H^4, H^2 \times H^2, \mathbf{CH}^2$.

Part 2. of the proposition follows.

- 3. Suppose that zero is not in the spectrum of \tilde{X} . From Properties 7 and 9, $\chi(X) = \tau(X) = 0$. From the classification of complex surfaces, X has a geometric structure [32, p. 148–149]. This contradicts part 2. of the proposition. \square

5.4 MORE DIMENSIONS

In this subsection we give some partial positive results about the zero-in-the-spectrum question for covers of compact manifolds of arbitrary dimension. Let us first recall some facts about index theory [18]. Let X be a closed Riemannian manifold. If $\dim(X)$ is even, consider the operator $d + d^*$ on $\Lambda^*(X)$. Give $\Lambda^*(X)$ the \mathbf{Z}_2 -grading coming from (3.12). Then the signature $\tau(X)$ equals the index of $d + d^*$. To say this in a more complicated way, the operator $d + d^*$ defines an element $[d + d^*]$ of the K-homology group $K_0(X)$. Let $\eta : X \rightarrow \text{pt.}$ be the (only) map from X to a point. Then $\eta_*([d + d^*]) \in K_0(\text{pt.})$. There is a map $A : K_0(\text{pt.}) \rightarrow K_0(\mathbf{C})$ which is the identity, as both sides are \mathbf{Z} . So we can say that $\tau(X) = A(\eta_*([d + d^*])) \in K_0(\mathbf{C})$.

We now extend the preceding remarks to the case of a group action. Let M be a normal cover of X with covering group Γ . The fiber bundle $M \rightarrow X$ is classified by a map $\nu : X \rightarrow B\Gamma$, defined up to homotopy. Let \tilde{d} be exterior differentiation on M . Consider the operator $\tilde{d} + \tilde{d}^*$. Taking into account the action of Γ on M , one can define a refined index $\text{ind}(\tilde{d} + \tilde{d}^*) \in K_0(C_r^*\Gamma)$, where $C_r^*\Gamma$ is the reduced group C^* -algebra of Γ .

We recall the statement of the Strong Novikov Conjecture (SNC) [18, 29]. This is a conjecture about a countable discrete group Γ , namely that the assembly map $A : K_*(B\Gamma) \rightarrow K_*(C_r^*\Gamma)$ is rationally injective. Many groups of a geometric origin, such as discrete subgroups of connected Lie groups or Gromov-hyperbolic groups, are known to satisfy SNC. There are no known groups which do not satisfy SNC.

PROPOSITION 19. *Let X be a closed Riemannian manifold with a surjective homomorphism $\pi_1(X) \rightarrow \Gamma$. Let M be the induced normal Γ -cover of X . Suppose that Γ satisfies SNC. Let $L(X) \in H^*(X; \mathbf{C})$ be the Hirzebruch L -class of X and let $*L(X) \in H_*(X; \mathbf{C})$ be its Poincaré dual. Then if $\nu_*(L(X)) \neq 0$ in $H_*(B\Gamma; \mathbf{C})$, zero lies in the spectrum of M . In fact, $0 \in \sigma\left(\Delta_{\frac{\dim(X)}{2}}\right)$ if $\dim(X)$ is even and $0 \in \sigma\left(\Delta_{\frac{\dim(X) \pm 1}{2}}\right)$ if $\dim(X)$ is odd.*

Proof. Suppose first that $\dim(X)$ is even. Suppose that zero does not lie in the spectrum of M . Then the operator $\tilde{d} + \tilde{d}^*$ is invertible. (More precisely, it is invertible as an operator on a Hilbert $C_r^*\Gamma$ -module of differential forms on M .) This implies that $\text{ind}(\tilde{d} + \tilde{d}^*)$ vanishes in $K_0(C_r^*\Gamma)$.

The higher index theorem says that

$$(5.10) \quad \text{ind}(\tilde{d} + \tilde{d}^*) = A(\nu_*([d + d^*])).$$

Let $A_{\mathbf{C}} : K_0(B\Gamma) \otimes \mathbf{C} \rightarrow K_0(C_r^*\Gamma) \otimes \mathbf{C}$ be the complexified assembly map. Using the isomorphism $K_0(B\Gamma) \otimes \mathbf{C} \cong H_{\text{even}}(B\Gamma; \mathbf{C})$, the higher index theorem implies that in $K_0(C_r^*\Gamma) \otimes \mathbf{C}$,

$$(5.11) \quad \text{ind}(\tilde{d} + \tilde{d}^*)_{\mathbf{C}} = A_{\mathbf{C}}(\nu_*(L(X))).$$

By assumption, $A_{\mathbf{C}}$ is injective. This gives a contradiction.

Let T be the operator obtained by restricting $\tilde{d} + \tilde{d}^*$ to

$$\Lambda^{\frac{\dim(X)}{2}}(M) \oplus \overline{\tilde{d}\Lambda^{\frac{\dim(X)}{2}}(M)} \oplus \overline{*d\Lambda^{\frac{\dim(X)}{2}}(M)}.$$

One can show that the other differential forms on M cancel out when computing the rational index of $\tilde{d} + \tilde{d}^*$, so T will have the same index as $\tilde{d} + \tilde{d}^*$. Then the same arguments apply to T to give $0 \in \sigma\left(\Delta_{\frac{\dim(X)}{2}}\right)$.

If $\dim(X)$ is odd, consider the even-dimensional manifold $X' = X \times S^1$ and the group $\Gamma' = \Gamma \times \mathbf{Z}$. As the proposition holds for X' , it must also hold for X . \square

COROLLARY 4. *Let X be a closed Riemannian manifold. Let $[X] \in H_{\dim(X)}(X; \mathbf{C})$ be its fundamental class. Suppose that there is a surjective homomorphism $\pi_1(X) \rightarrow \Gamma$ such that Γ satisfies SNC and the composite map $X \rightarrow B\pi_1(X) \rightarrow B\Gamma$ sends $[X]$ to a nonzero element of $H_{\dim(X)}(B\Gamma; \mathbf{C})$. Let M be the induced normal Γ -cover of X . Then on M , $0 \in \sigma\left(\Delta_{\frac{\dim(X)}{2}}\right)$ if $\dim(X)$ is even and $0 \in \sigma\left(\Delta_{\frac{\dim(X) \pm 1}{2}}\right)$ if $\dim(X)$ is odd.*

Proof. As the Hirzebruch L -class starts out as $L(X) = 1 + \dots$, its Poincaré dual is of the form $*L(X) = \dots + [X]$. The corollary follows from Proposition 19. \square

COROLLARY 5. *Let X be a closed aspherical Riemannian manifold whose fundamental group satisfies SNC. Then on \tilde{X} , $0 \in \sigma\left(\Delta_{\frac{\dim(X)}{2}}\right)$ if $\dim(X)$ is even and $0 \in \sigma\left(\Delta_{\frac{\dim(X) \pm 1}{2}}\right)$ if $\dim(X)$ is odd.*

Proof. This follows from Corollary 4. \square

EXAMPLES.

1. If $X = T^n$ then Corollary 5 is consistent with Example 2 of Section 2.
2. If X is a compact quotient of H^{2n} then Corollary 5 is consistent with Example 3 of Section 2.
3. If X is a compact quotient of H^{2n+1} then Corollary 5 is consistent with Example 4 of Section 2.
4. If X is a closed nonpositively-curved locally symmetric space then Corollary 5 is consistent with the second remark after Proposition 7.

If X is a closed aspherical manifold, it is known that SNC implies that the rational Pontryagin classes of X are homotopy-invariants [18] and that X does not admit a Riemannian metric of positive scalar curvature [29]. Thus we see that these three questions about aspherical manifolds, namely homotopy-invariance of rational Pontryagin classes, (non)existence of positive-scalar-curvature metrics and the zero-in-the-spectrum question, are roughly all on the same footing.

If X is a closed aspherical Riemannian manifold, one can ask for which p one has $0 \in \sigma(\Delta_p)$ on \tilde{X} . The case of locally symmetric spaces is covered by the second remark after Proposition 7. Another interesting class of aspherical manifolds consists of those with amenable fundamental group. By [5], $\text{Ker}(\Delta_p) = 0$ for all p . By Corollary 3, $0 \in \sigma(\Delta_0)$.

PROPOSITION 20. *If X is a closed aspherical manifold such that $\pi_1(X)$ has a nilpotent subgroup of finite index then $0 \in \sigma(\Delta_p)$ on \tilde{X} for all $p \in [0, \dim(X)]$.*

Proof. First, X is homotopy-equivalent to an infranilmanifold, that is, a quotient of the form $\Gamma \backslash G/K$ where K is a finite group, G is the

semidirect product of K and a connected simply-connected nilpotent Lie group and Γ is a discrete cocompact subgroup of G [12, Theorem 6.4]. We may as well assume that $X = \Gamma \backslash G/K$. By passing to a finite cover, we may assume that K is trivial. That is, X is a nilmanifold. From [27, Corollary 7.28], $H^p(X; \mathbb{C}) \cong H^p(\mathfrak{g}, \mathbb{C})$, the Lie algebra cohomology of \mathfrak{g} . From [7], $H^p(\mathfrak{g}, \mathbb{C}) \neq 0$ for all $p \in [0, \dim(X)]$. Thus for all $p \in [0, \dim(X)]$, $H^p(X; \mathbb{C}) \neq 0$.

Now let ω be a nonzero harmonic p -form on X . Let $\pi^*\omega$ be its pullback to \tilde{X} . The idea is to construct low-energy square-integrable p -forms on X by multiplying $\pi^*\omega$ by appropriate functions on X . We define the functions as in [2, §2]. Take a smooth triangulation of X and choose a fundamental domain F for the lifted triangulation of \tilde{X} . If E is a finite subset of $\pi_1(X)$, let χ_H be the characteristic function of $H = \bigcup_{g \in E} g \cdot F$. Given numbers $0 < \epsilon_1 < \epsilon_2 < 1$, choose a nonincreasing function $\psi \in C_0^\infty([0, \infty))$ which is identically one on $[0, \epsilon_1]$ and identically zero on $[\epsilon_2, \infty)$. Define a compactly-supported function f_E on \tilde{X} by $f_E(m) = \psi(d(m, H))$. Then there is a constant $C_1 > 0$, independent of E , such that

$$(5.12) \quad \int_{\tilde{X}} |df_E|^2 \leq C_1 \text{area}(\partial H).$$

Define $\rho_E \in \Lambda^p(\tilde{X})$ by $\rho_E = f_E \cdot \pi^*\omega$. We have $d\rho_E = df_E \wedge \pi^*\omega$ and $d^*\rho_E = -i(df_E)\pi^*\omega$. As f_E is identically one on H , it follows that there is a constant $C > 0$, independent of E , such that

$$(5.13) \quad \frac{\int_{\tilde{X}} [|d\rho_E|^2 + |d^*\rho_E|^2]}{\int_{\tilde{X}} |\rho_E|^2} \leq C \frac{\text{area}(\partial H)}{\text{vol}(H)}.$$

As $\pi_1(X)$ is amenable, by an appropriate choice of E this ratio can be made arbitrarily small. Thus $0 \in \sigma(\Delta_p)$. \square

QUESTION. Does the conclusion of Proposition 20 hold if we only assume that $\pi_1(X)$ is amenable?

6. TOPOLOGICALLY TAME MANIFOLDS

Another class of manifolds for which one can hope to get some nontrivial results about the zero-in-the-spectrum question is given by *topologically tame* manifolds, meaning manifolds M which are diffeomorphic to the interior of a compact manifold N with boundary. If M has finite volume then $\text{Ker}(\Delta_0) \neq 0$,