

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 42 (1996)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE ZERO-IN-THE-SPECTRUM QUESTION
Autor: LOTT, John
Kapitel: 5.3 Four Dimensions
DOI: <https://doi.org/10.5169/seals-87882>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 19.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Now let Y be a 3-manifold satisfying the conditions of Proposition 13. If $\partial Y \neq \emptyset$, we define Δ_p on \tilde{Y} using absolute boundary conditions on $\partial \tilde{Y}$.

PROPOSITION 16. *Zero lies in the spectrum of \tilde{Y} .*

Proof. This is a consequence of Propositions 11 and 13. \square

5.3 FOUR DIMENSIONS

In this subsection we relate the zero-in-the-spectrum question to a question about Euler characteristics of closed 4-dimensional manifolds.

If M is a Riemannian 4-manifold then the Hodge decomposition gives

$$\begin{aligned}
 (5.7) \quad \Lambda^0(M) &= \text{Ker}(\Delta_0) \oplus \Lambda^0(M)/\text{Ker}(d), \\
 \Lambda^1(M) &= \text{Ker}(\Delta_1) \oplus \overline{d\Lambda^0(M)} \oplus \Lambda^1(M)/\text{Ker}(d), \\
 \Lambda^2(M) &= \text{Ker}(\Delta_2) \oplus \overline{d\Lambda^1(M)} \oplus \overline{*d\Lambda^1(M)}, \\
 \Lambda^3(M) &= *\text{Ker}(\Delta_1) \oplus \overline{*d\Lambda^0(M)} \oplus *(\Lambda^1(M)/\text{Ker}(d)), \\
 \Lambda^4(M) &= *\text{Ker}(\Delta_0) \oplus *(\Lambda^0(M)/\text{Ker}(d)).
 \end{aligned}$$

Thus for the zero-in-the-spectrum question, it is enough to consider $\text{Ker}(\Delta_0)$, $\text{Ker}(\Delta_1)$, $\sigma(\Delta_0 \text{ on } \Lambda^0/\text{Ker}(d))$, $\sigma(\Delta_1 \text{ on } \Lambda^1/\text{Ker}(d))$ and $\text{Ker}(\Delta_2)$.

Let Γ be a finitely-presented group. Recall that Γ is the fundamental group of some closed 4-manifold. To see this, take a finite presentation of Γ . Embed the resulting presentation complex in \mathbf{R}^5 and take the boundary of a regular neighborhood to get the manifold.

Now consider the Euler characteristics of all closed 4-manifolds X with fundamental group Γ . Given X , we have $\chi(X \# \mathbf{CP}^2) = \chi(X) + 1$. Thus it is easy to make the Euler characteristic big. However, it is not so easy to make it small. From what has been said,

$$\begin{aligned}
 &\{\chi(X): X \text{ is a closed connected oriented 4-manifold with} \\
 (5.8) \quad &\pi_1(X) = \Gamma\} = \{n \in \mathbf{Z} : n \geq q(\Gamma)\}
 \end{aligned}$$

for some $q(\Gamma)$. *A priori* $q(\Gamma) \in \mathbf{Z} \cup \{-\infty\}$, but in fact $q(\Gamma) \in \mathbf{Z}$ [17, Theorem 1]. (This also follows from (5.9) below.) It is a basic problem in 4-manifold topology to get good estimates of $q(\Gamma)$.

Suppose that $\pi_1(X) = \Gamma$. From Properties 4, 7 and 8 above,

$$(5.9) \quad \chi(X) = 2b_0^{(2)}(\Gamma) - 2b_1^{(2)}(\Gamma) + b_2^{(2)}(X).$$

In particular, if $b_1^{(2)}(\Gamma) = 0$ then $\chi(X) \geq 0$ and so $q(\Gamma) \geq 0$. This is the case, for example, when Γ is big or when Γ is amenable [5].

PROPOSITION 17. *Let X be a closed 4-manifold. Then zero is not in the spectrum of \tilde{X} if and only if $\pi_1(X)$ is big and $\chi(X) = 0$.*

Proof. Suppose that zero is not in the spectrum of \tilde{X} . Then from Proposition 11, $\pi_1(X)$ must be big. Furthermore, $\text{Ker}(\Delta_2) = 0$. From Property 1 and (5.9), $\chi(X) = 0$.

Now suppose that $\pi_1(X)$ is big and $\chi(X) = 0$. From Proposition 11, $0 \notin \sigma(\Delta_0)$ and $0 \notin \sigma(\Delta_1)$. From Property 1 and (5.9), $\text{Ker}(\Delta_2) = 0$. Then from (5.7), zero is not in the spectrum of \tilde{X} . \square

REMARK. If zero is not in the spectrum of \tilde{X} then it follows from Property 9 that in addition, $\tau(X) = 0$. Also, as will be explained later in Corollary 4, if $\pi_1(X)$ satisfies the Strong Novikov Conjecture then $\nu_*([X])$ vanishes in $H_4(B\pi_1(X); \mathbb{C})$.

In summary, we have shown that the answer to the zero-in-the-spectrum question is “yes” for universal covers of closed 4-manifolds if and only if the following conjecture is true.

CONJECTURE 2. *If Γ is a big group then $q(\Gamma) > 0$.*

We now give some partial positive results on the zero-in-the-spectrum question for universal covers of closed 4-manifolds. Recall that there is a notion, due to Thurston, of a manifold having a geometric structure. This is especially important for 3-manifolds. The 4-manifolds with geometric structures have been studied by Wall [32].

PROPOSITION 18. *Let X be a closed 4-manifold. Then zero is in the spectrum of \tilde{X} if*

1. $\pi_1(X)$ has property T or
2. X has a geometric structure (and an arbitrary Riemannian metric) or
3. X has a complex structure (and an arbitrary Riemannian metric).

Proof.

1. If X has property T then the ordinary first Betti number of X vanishes [6]. Thus $\chi(X) = 2 + b_2(X) > 0$. Part 1. of the proposition follows.
2. The geometries of [32] all fall into at least one of the following classes :

- a. $b_0^{(2)} \neq 0 : S^4, S^2 \times S^2, \mathbf{CP}^2$.
- b. $0 \in \sigma(\Delta_0 \text{ on } \Lambda^0 / \text{Ker}(d)) : \mathbf{R}^4, S^3 \times \mathbf{R}, S^2 \times \mathbf{R}^2, Nil^3 \times \mathbf{R}, Nil^4, Sol_0^4, Sol_1^4, Sol_{m,n}^4$.
- c. $b_1^{(2)} \neq 0 : S^2 \times H^2$.
- d. $0 \in \sigma(\Delta_1 \text{ on } \Lambda^1 / \text{Ker}(d)) : H^3 \times \mathbf{R}, \widetilde{SL_2} \times \mathbf{R}, H^2 \times \mathbf{R}^2$.
- e. $\chi > 0 : H^4, H^2 \times H^2, \mathbf{CH}^2$.

Part 2. of the proposition follows.

- 3. Suppose that zero is not in the spectrum of \tilde{X} . From Properties 7 and 9, $\chi(X) = \tau(X) = 0$. From the classification of complex surfaces, X has a geometric structure [32, p. 148–149]. This contradicts part 2. of the proposition. \square

5.4 MORE DIMENSIONS

In this subsection we give some partial positive results about the zero-in-the-spectrum question for covers of compact manifolds of arbitrary dimension. Let us first recall some facts about index theory [18]. Let X be a closed Riemannian manifold. If $\dim(X)$ is even, consider the operator $d + d^*$ on $\Lambda^*(X)$. Give $\Lambda^*(X)$ the \mathbf{Z}_2 -grading coming from (3.12). Then the signature $\tau(X)$ equals the index of $d + d^*$. To say this in a more complicated way, the operator $d + d^*$ defines an element $[d + d^*]$ of the K-homology group $K_0(X)$. Let $\eta : X \rightarrow \text{pt.}$ be the (only) map from X to a point. Then $\eta_*([d + d^*]) \in K_0(\text{pt.})$. There is a map $A : K_0(\text{pt.}) \rightarrow K_0(\mathbf{C})$ which is the identity, as both sides are \mathbf{Z} . So we can say that $\tau(X) = A(\eta_*([d + d^*])) \in K_0(\mathbf{C})$.

We now extend the preceding remarks to the case of a group action. Let M be a normal cover of X with covering group Γ . The fiber bundle $M \rightarrow X$ is classified by a map $\nu : X \rightarrow B\Gamma$, defined up to homotopy. Let \tilde{d} be exterior differentiation on M . Consider the operator $\tilde{d} + \tilde{d}^*$. Taking into account the action of Γ on M , one can define a refined index $\text{ind}(\tilde{d} + \tilde{d}^*) \in K_0(C_r^*\Gamma)$, where $C_r^*\Gamma$ is the reduced group C^* -algebra of Γ .

We recall the statement of the Strong Novikov Conjecture (SNC) [18, 29]. This is a conjecture about a countable discrete group Γ , namely that the assembly map $A : K_*(B\Gamma) \rightarrow K_*(C_r^*\Gamma)$ is rationally injective. Many groups of a geometric origin, such as discrete subgroups of connected Lie groups or Gromov-hyperbolic groups, are known to satisfy SNC. There are no known groups which do not satisfy SNC.