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Now let  $Y$  be a 3-manifold satisfying the conditions of Proposition 13. If  $\partial Y \neq \emptyset$ , we define  $\Delta_p$  on  $\tilde{Y}$  using absolute boundary conditions on  $\partial \tilde{Y}$ .

PROPOSITION 16. *Zero lies in the spectrum of  $\tilde{Y}$ .*

*Proof.* This is a consequence of Propositions 11 and 13.  $\square$

### 5.3 FOUR DIMENSIONS

In this subsection we relate the zero-in-the-spectrum question to a question about Euler characteristics of closed 4-dimensional manifolds.

If  $M$  is a Riemannian 4-manifold then the Hodge decomposition gives

$$(5.7) \quad \begin{aligned} \Lambda^0(M) &= \text{Ker}(\Delta_0) \oplus \Lambda^0(M)/\text{Ker}(d), \\ \Lambda^1(M) &= \text{Ker}(\Delta_1) \oplus \overline{d\Lambda^0(M)} \oplus \Lambda^1(M)/\text{Ker}(d), \\ \Lambda^2(M) &= \text{Ker}(\Delta_2) \oplus \overline{d\Lambda^1(M)} \oplus \overline{*d\Lambda^1(M)}, \\ \Lambda^3(M) &= * \text{Ker}(\Delta_1) \oplus * \overline{d\Lambda^0(M)} \oplus *(\Lambda^1(M)/\text{Ker}(d)), \\ \Lambda^4(M) &= * \text{Ker}(\Delta_0) \oplus *(\Lambda^0(M)/\text{Ker}(d)). \end{aligned}$$

Thus for the zero-in-the-spectrum question, it is enough to consider  $\text{Ker}(\Delta_0)$ ,  $\text{Ker}(\Delta_1)$ ,  $\sigma(\Delta_0 \text{ on } \Lambda^0/\text{Ker}(d))$ ,  $\sigma(\Delta_1 \text{ on } \Lambda^1/\text{Ker}(d))$  and  $\text{Ker}(\Delta_2)$ .

Let  $\Gamma$  be a finitely-presented group. Recall that  $\Gamma$  is the fundamental group of some closed 4-manifold. To see this, take a finite presentation of  $\Gamma$ . Embed the resulting presentation complex in  $\mathbf{R}^5$  and take the boundary of a regular neighborhood to get the manifold.

Now consider the Euler characteristics of all closed 4-manifolds  $X$  with fundamental group  $\Gamma$ . Given  $X$ , we have  $\chi(X \# \mathbf{CP}^2) = \chi(X) + 1$ . Thus it is easy to make the Euler characteristic big. However, it is not so easy to make it small. From what has been said,

$$(5.8) \quad \begin{aligned} \{\chi(X) : X \text{ is a closed connected oriented 4-manifold with} \\ \pi_1(X) = \Gamma\} = \{n \in \mathbf{Z} : n \geq q(\Gamma)\} \end{aligned}$$

for some  $q(\Gamma)$ . *A priori*  $q(\Gamma) \in \mathbf{Z} \cup \{-\infty\}$ , but in fact  $q(\Gamma) \in \mathbf{Z}$  [17, Theorem 1]. (This also follows from (5.9) below.) It is a basic problem in 4-manifold topology to get good estimates of  $q(\Gamma)$ .

Suppose that  $\pi_1(X) = \Gamma$ . From Properties 4, 7 and 8 above,

$$(5.9) \quad \chi(X) = 2b_0^{(2)}(\Gamma) - 2b_1^{(2)}(\Gamma) + b_2^{(2)}(X).$$

In particular, if  $b_1^{(2)}(\Gamma) = 0$  then  $\chi(X) \geq 0$  and so  $q(\Gamma) \geq 0$ . This is the case, for example, when  $\Gamma$  is big or when  $\Gamma$  is amenable [5].

PROPOSITION 17. *Let  $X$  be a closed 4-manifold. Then zero is not in the spectrum of  $\widetilde{X}$  if and only if  $\pi_1(X)$  is big and  $\chi(X) = 0$ .*

*Proof.* Suppose that zero is not in the spectrum of  $\widetilde{X}$ . Then from Proposition 11,  $\pi_1(X)$  must be big. Furthermore,  $\text{Ker}(\Delta_2) = 0$ . From Property 1 and (5.9),  $\chi(X) = 0$ .

Now suppose that  $\pi_1(X)$  is big and  $\chi(X) = 0$ . From Proposition 11,  $0 \notin \sigma(\Delta_0)$  and  $0 \notin \sigma(\Delta_1)$ . From Property 1 and (5.9),  $\text{Ker}(\Delta_2) = 0$ . Then from (5.7), zero is not in the spectrum of  $\widetilde{X}$ .  $\square$

REMARK. If zero is not in the spectrum of  $\widetilde{X}$  then it follows from Property 9 that in addition,  $\tau(X) = 0$ . Also, as will be explained later in Corollary 4, if  $\pi_1(X)$  satisfies the Strong Novikov Conjecture then  $\nu_*([X])$  vanishes in  $H_4(B\pi_1(X); \mathbf{C})$ .

In summary, we have shown that the answer to the zero-in-the-spectrum question is “yes” for universal covers of closed 4-manifolds if and only if the following conjecture is true.

CONJECTURE 2. *If  $\Gamma$  is a big group then  $q(\Gamma) > 0$ .*

We now give some partial positive results on the zero-in-the-spectrum question for universal covers of closed 4-manifolds. Recall that there is a notion, due to Thurston, of a manifold having a geometric structure. This is especially important for 3-manifolds. The 4-manifolds with geometric structures have been studied by Wall [32].

PROPOSITION 18. *Let  $X$  be a closed 4-manifold. Then zero is in the spectrum of  $\widetilde{X}$  if*

1.  $\pi_1(X)$  has property T or
2.  $X$  has a geometric structure (and an arbitrary Riemannian metric) or
3.  $X$  has a complex structure (and an arbitrary Riemannian metric).

*Proof.*

1. If  $X$  has property T then the ordinary first Betti number of  $X$  vanishes [6]. Thus  $\chi(X) = 2 + b_2(X) > 0$ . Part 1. of the proposition follows.
2. The geometries of [32] all fall into at least one of the following classes :

- a.  $b_0^{(2)} \neq 0 : S^4, S^2 \times S^2, \mathbf{CP}^2$ .
- b.  $0 \in \sigma(\Delta_0 \text{ on } \Lambda^0 / \text{Ker}(d)) : \mathbf{R}^4, S^3 \times \mathbf{R}, S^2 \times \mathbf{R}^2, \text{Nil}^3 \times \mathbf{R}, \text{Nil}^4, \text{Sol}_0^4, \text{Sol}_1^4, \text{Sol}_{m,n}^4$ .
- c.  $b_1^{(2)} \neq 0 : S^2 \times H^2$ .
- d.  $0 \in \sigma(\Delta_1 \text{ on } \Lambda^1 / \text{Ker}(d)) : H^3 \times \mathbf{R}, \widetilde{SL}_2 \times \mathbf{R}, H^2 \times \mathbf{R}^2$ .
- e.  $\chi > 0 : H^4, H^2 \times H^2, \mathbf{CH}^2$ .

Part 2. of the proposition follows.

- 3. Suppose that zero is not in the spectrum of  $\tilde{X}$ . From Properties 7 and 9,  $\chi(X) = \tau(X) = 0$ . From the classification of complex surfaces [32, p. 148–149]. This contradicts part 2. of the proposition.  $\square$

## 5.4 MORE DIMENSIONS

In this subsection we give some partial positive results about the zero-in-the-spectrum question for covers of compact manifolds of arbitrary dimension. Let us first recall some facts about index theory [18]. Let  $X$  be a closed Riemannian manifold. If  $\dim(X)$  is even, consider the operator  $d + d^*$  on  $\Lambda^*(X)$ . Give  $\Lambda^*(X)$  the  $\mathbf{Z}_2$ -grading coming from (3.12). Then the signature  $\tau(X)$  equals the index of  $d + d^*$ . To say this in a more complicated way, the operator  $d + d^*$  defines an element  $[d + d^*]$  of the K-homology group  $K_0(X)$ . Let  $\eta : X \rightarrow \text{pt.}$  be the (only) map from  $X$  to a point. Then  $\eta_*([d + d^*]) \in K_0(\text{pt.})$ . There is a map  $A : K_0(\text{pt.}) \rightarrow K_0(\mathbf{C})$  which is the identity, as both sides are  $\mathbf{Z}$ . So we can say that  $\tau(X) = A(\eta([d + d^*])) \in K_0(\mathbf{C})$ .

We now extend the preceding remarks to the case of a group action. Let  $M$  be a normal cover of  $X$  with covering group  $\Gamma$ . The fiber bundle  $M \rightarrow X$  is classified by a map  $\nu : X \rightarrow B\Gamma$ , defined up to homotopy. Let  $\tilde{d}$  be exterior differentiation on  $M$ . Consider the operator  $\tilde{d} + \tilde{d}^*$ . Taking into account the action of  $\Gamma$  on  $M$ , one can define a refined index  $\text{ind}(\tilde{d} + \tilde{d}^*) \in K_0(C_r^*\Gamma)$ , where  $C_r^*\Gamma$  is the reduced group  $C^*$ -algebra of  $\Gamma$ .

We recall the statement of the Strong Novikov Conjecture (SNC) [18, 29]. This is a conjecture about a countable discrete group  $\Gamma$ , namely that the assembly map  $A : K_*(B\Gamma) \rightarrow K_*(C_r^*\Gamma)$  is rationally injective. Many groups of a geometric origin, such as discrete subgroups of connected Lie groups or Gromov-hyperbolic groups, are known to satisfy SNC. There are no known groups which do not satisfy SNC.