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**Autor:** LOTT, John  
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In what follows,  $\Gamma$  will denote a finitely-presented group. Given a presentation of  $\Gamma$ , there is an associated 2-dimensional CW-complex  $K$  which we call the *presentation complex*. To form it, make a bouquet of circles indexed by the generators of  $\Gamma$ . Attach 2-cells based on the relations of  $\Gamma$ . (We allow trivial or repeated relations in the presentation.) This is the presentation complex.

DEFINITION 7. Put  $b_0^{(2)}(\Gamma) = b_0^{(2)}(K)$ ,  $b_1^{(2)}(\Gamma) = b_1^{(2)}(K)$ ,  $\alpha_1(\Gamma) = \alpha_1(K)$  and  $\alpha_2(\Gamma) = \alpha_2(K)$ .

By Property 4 above, Definition 7 makes sense in that the choice of presentation of  $\Gamma$  does not matter.

As the invariants  $b_0^{(2)}(\Gamma)$ ,  $b_1^{(2)}(\Gamma)$ ,  $\alpha_1(\Gamma)$  and  $\alpha_2(\Gamma)$  will play an important role, let us state explicitly what they measure. First, from Property 5,  $b_0^{(2)}(\Gamma)$  tells us whether or not  $\Gamma$  is infinite. In general,  $b_0^{(2)}(\Gamma) = \frac{1}{|\Gamma|}$ . Next, from Property 1,  $b_1^{(2)}(\Gamma)$  tells us whether or not  $M$  has square-integrable harmonic 1-forms (or  $\tilde{K}$  has square-integrable harmonic 1-cochains). From Property 2,  $\alpha_1(\Gamma)$  tells us whether or not the Laplacian  $\Delta_0$ , acting on functions on  $M$ , has a gap in its spectrum away from zero. In fact, Property 6 is just a restatement of Corollary 3. Finally, from Property 2,  $\alpha_2(\Gamma)$  tells us whether or not the spectrum of the Laplacian on  $\Lambda^1(M)/\text{Ker}(d)$  goes down to zero.

## 5.1 BIG AND SMALL GROUPS

Let us first introduce a convenient terminology for the purposes of the present paper.

DEFINITION 8. *The group  $\Gamma$  is big if it is nonamenable,  $b_1^{(2)}(\Gamma) = 0$  and  $\alpha_2(\Gamma) = \infty^+$ . Otherwise,  $\Gamma$  is small.*

We recall that  $\Delta_p$  denotes the Laplace-Beltrami operator on the universal cover  $M$ .

PROPOSITION 11. *Let  $X$  and  $M$  be as above. The group  $\pi_1(X)$  is small if and only if  $0 \in \sigma(\Delta_0)$  or  $0 \in \sigma(\Delta_1)$ .*

*Proof.* This follows immediately from Properties 1, 2, 4, 5 and 6 above.  $\square$

The question arises as to which groups are big and which are small. Clearly any amenable group is small.

PROPOSITION 12. *Fundamental groups of compact surfaces are small.*

*Proof.* Suppose that  $\Sigma$  is a compact surface and  $\Gamma = \pi_1(\Sigma)$ . If  $\Sigma$  has boundary then  $\Gamma$  is a free group  $F_j$  on some number  $j$  of generators. If  $j = 0$  or  $j = 1$  then  $\Gamma$  is amenable. If  $j > 1$  then  $b_1^{(2)}(\Gamma) = j - 1 > 0$ .

Suppose now that  $\Sigma$  is closed. If  $\chi(\Sigma) \geq 0$  then  $\Gamma$  is amenable. If  $\chi(\Sigma) < 0$  then  $b_1^{(2)}(\Gamma) = -\chi(\Sigma) > 0$ .  $\square$

We now extend Proposition 12 to 3-manifold groups. We use some facts about compact connected 3-manifolds  $Y$ , possibly with boundary. (See, for example, [21, Section 6]). Again, all of our manifolds are assumed to be oriented. First,  $Y$  has a decomposition as a connected sum  $Y = Y_1 \# Y_2 \# \dots \# Y_r$  of *prime* 3-manifolds. A prime 3-manifold is *exceptional* if it is closed and no finite cover of it is homotopy-equivalent to a Seifert, Haken or hyperbolic 3-manifold. No exceptional prime 3-manifolds are known and it is likely that there are none.

PROPOSITION 13 (Lott-Lück). *Suppose that  $Y$  is a compact connected oriented 3-manifold, possibly with boundary, none of whose prime factors are exceptional. Then  $\pi_1(Y)$  is small.*

*Proof.* We argue by contradiction. Suppose that  $\pi_1(Y)$  is big. First,  $\pi_1(Y)$  must be infinite. If  $\partial Y$  has any connected components which are 2-spheres then we can cap them off with 3-balls without changing  $\pi_1(Y)$ . So we can assume that  $\partial Y$  does not have any 2-sphere components. In particular,  $\chi(Y) = \frac{1}{2}\chi(\partial Y) \leq 0$ . From [21, Theorem 0.1.1],

$$(5.3) \quad b_1^{(2)}(Y) = (r - 1) - \sum_{i=1}^r \frac{1}{|\pi_1(Y_i)|} - \chi(Y).$$

As this must vanish, we have  $\chi(Y) = 0$  and either

1.  $\{|\pi_1(Y_i)|\}_{i=1}^r = \{2, 2, 1, \dots, 1\}$  or
2.  $\{|\pi_1(Y_i)|\}_{i=1}^r = \{\infty, 1, \dots, 1\}$ .

It follows that  $\partial Y$  is empty or a disjoint union of 2-tori. As there are no 2-spheres in  $\partial Y$ , if  $|\pi_1(Y_i)| = 1$  then  $Y_i$  is a homotopy 3-sphere. Thus  $Y$  is homotopy-equivalent to either

1.  $\mathbf{RP}^3 \# \mathbf{RP}^3$  or
2. A prime 3-manifold  $Y'$  with infinite fundamental group whose boundary is empty or a disjoint union of 2-tori.

If  $Y$  is homotopy-equivalent to  $\mathbf{R}P^3\#\mathbf{R}P^3$  then  $\pi_1(Y)$  is amenable, which is a contradiction. So we must be in the second case. Using Property 3, we may assume that  $Y = Y'$ . Then as  $Y$  is prime, it follows from [24, Chapter 1] that either  $Y = S^1 \times D^2$  or  $Y$  has incompressible (or empty) boundary. If  $Y = S^1 \times D^2$  then  $\pi_1(Y)$  is amenable. If  $Y$  has incompressible (or empty) boundary then from [21, Theorem 0.1.5],  $\alpha_2(Y) \leq 2$  unless  $Y$  is a closed 3-manifold with an  $\mathbf{R}^3$ ,  $\mathbf{R} \times S^2$  or  $Sol$  geometric structure. In the latter cases,  $\Gamma$  is amenable. Thus in any case, we get a contradiction.  $\square$

The next proposition gives examples of big groups.

PROPOSITION 14.

1. *A product of two nonamenable groups is big.*
2. *If  $Y$  is a closed nonpositively-curved locally symmetric space of dimension greater than three, with no Euclidean factors in  $\tilde{Y}$ , then  $\pi_1(Y)$  is big.*

*Proof.* 1. Suppose that  $\Gamma = \Gamma_1 \times \Gamma_2$  with  $\Gamma_1$  and  $\Gamma_2$  nonamenable. Then  $\Gamma$  is nonamenable. Let  $K_1$  and  $K_2$  be presentation complexes with fundamental groups  $\Gamma_1$  and  $\Gamma_2$ , respectively. Put  $K = K_1 \times K_2$ . Then  $\Gamma = \pi_1(K)$ . Let  $\Delta_p(\tilde{K})$ ,  $\Delta_p(\tilde{K}_1)$  and  $\Delta_p(\tilde{K}_2)$  denote the Laplace-Beltrami operator on  $p$ -cochains on  $\tilde{K}$ ,  $\tilde{K}_1$  and  $\tilde{K}_2$ , respectively, as defined in Subsection 5.2 below. Then

$$(5.4) \quad \inf(\sigma(\Delta_1(\tilde{K}))) = \min(\inf(\sigma(\Delta_1(\tilde{K}_1))) + \inf(\sigma(\Delta_0(\tilde{K}_2))), \inf(\sigma(\Delta_0(\tilde{K}_1))) + \inf(\sigma(\Delta_1(\tilde{K}_2)))) > 0.$$

Using Proposition 11, the first part of the proposition follows.

2. If  $\tilde{Y}$  is irreducible then part 2. of the proposition follows from the second remark after Proposition 7. If  $\tilde{Y}$  is reducible then we can use an argument similar to (5.4).  $\square$

REMARK. Let  $\Gamma$  be an infinite finitely-presented discrete group with Kazhdan's property T. From [6, p. 47],  $H^1(\Gamma; \ell^2(\Gamma)) = 0$ . This implies that  $\Gamma$  is nonamenable and  $b_1^{(2)}(\Gamma) = 0$ . We do not know if it is necessarily true that  $\alpha_2(\Gamma) = \infty^+$ .

5.2 TWO AND THREE DIMENSIONS

In this subsection we relate the zero-in-the-spectrum question to a question in combinatorial group theory. Let  $K$  be a finite connected 2-dimensional