5. Universal Covers

Suppose that M is the universal cover of a compact Riemannian manifold X. We give M the pulled-back Riemannian metric and consider the Laplace-Beltrami operator $\triangle_p$ on M. There are numerical invariants derived from $\{\triangle_p\}_{p\geq 0}$, the so-called L^2-Betti numbers $\{b_p^{(2)}(X)\}_{p\geq 0}$ and Novikov-Shubin invariants $\{\alpha_{p+1}(X)\}_{p\geq 0}$. The L^2-Betti numbers lie in $[0,\infty)$ and the Novikov-Shubin invariants lie in $[0,\infty]\cup\infty^+$. Here ∞^+ is a formal symbol which should be considered to be greater than ∞. Roughly speaking, $b_p^{(2)}(X)$ measures the size of $\mathrm{Ker}(\triangle_p)$ as a $\pi_1(X)$-module and $\alpha_{p+1}(X)$ measures the thickness near zero of the spectrum of $\triangle_p$ on $\Lambda^p(M)/\mathrm{Ker}(d)$; the larger $\alpha_{p+1}(X)$, the thinner the spectrum near zero. We refer to [21, 22, 26] for the definitions of these invariants. We will only need the following properties:

Properties.

1. $b_p^{(2)}(X)=0 \iff \mathrm{Ker}(\triangle_p)=0$.
2. $0\notin\sigma\big(\triangle_p \text{ on } \Lambda^p(M)/\mathrm{Ker}(d)\big) \iff \alpha_{p+1}=\infty^+$.
3. $b_p^{(2)}(X)$ and $\alpha_p(X)$ are homotopy-invariants of X.
4. $b_0^{(2)}(X)$, $b_1^{(2)}(X)$, $\alpha_1(X)$ and $\alpha_2(X)$ only depend on $\pi_1(X)$.
5. $b_0^{(2)}(X)=0$ if and only if $\pi_1(X)$ is infinite.
6. $\alpha_1(X)=\infty^+$ if and only if $\pi_1(X)$ is finite or nonamenable.
7. The Euler characteristic of X satisfies

$$\chi(X)=\sum_p(-1)^p\, b_p^{(2)}(X) \tag{5.1}$$

8. If X^n is closed then $b_{n-p}^{(2)}(X)=b_p^{(2)}(X)$.
9. If X^{4k} is closed then there are nonnegative numbers $b_{2k,\pm}^{(2)}(X)$ such that $b_{2k}^{(2)}(X)=b_{2k,+}^{(2)}(X)+b_{2k,-}^{(2)}(X)$ and the signature of X satisfies

$$\tau(X)=b_{2k,+}^{(2)}(X)-b_{2k,-}^{(2)}(X). \tag{5.2}$$

One can extend properties 1-7 from compact Riemannian manifolds X to finite CW-complexes K.

In what follows, Γ will denote a finitely-presented group. Given a presentation of Γ, there is an associated 2-dimensional CW-complex K which we call the *presentation complex.* To form it, make a bouquet of circles indexed by the generators of Γ. Attach 2-cells based on the relations of Γ. (We allow trivial or repeated relations in the presentation.) This is the presentation complex.

DEFINITION 7. *Put* $b_0^{(2)}(\Gamma) = b_0^{(2)}(K)$, $b_1^{(2)}(\Gamma) = b_1^{(2)}(K)$, $\alpha_1(\Gamma) = \alpha_1(K)$ *and* $\alpha_2(\Gamma) = \alpha_2(K)$.

By Property 4 above, Definition 7 makes sense in that the choice of presentation of Γ does not matter.

As the invariants $b_0^{(2)}(\Gamma)$, $b_1^{(2)}(\Gamma)$, $\alpha_1(\Gamma)$ and $\alpha_2(\Gamma)$ will play an important role, let us state explicitly what they measure. First, from Property 5, $b_0^{(2)}(\Gamma)$ tells us whether or not Γ is infinite. In general, $b_0^{(2)}(\Gamma) = \frac{1}{|\Gamma|}$. Next, from Property 1, $b_1^{(2)}(\Gamma)$ tells us whether or not M has square-integrable harmonic 1-forms (or $\widetilde{K}$ has square-integrable harmonic 1-cochains). From Property 2, $\alpha_1(\Gamma)$ tells us whether or not the Laplacian $\triangle_0$, acting on functions on M, has a gap in its spectrum away from zero. In fact, Property 6 is just a restatement of Corollary 3. Finally, from Property 2, $\alpha_2(\Gamma)$ tells us whether or not the spectrum of the Laplacian on $\Lambda^1(M)/\operatorname{Ker}(d)$ goes down to zero.

5.1 BIG AND SMALL GROUPS

Let us first introduce a convenient terminology for the purposes of the present paper.

DEFINITION 8. *The group* Γ *is big if it is nonamenable,* $b_1^{(2)}(\Gamma) = 0$ *and* $\alpha_2(\Gamma) = \infty^+$. *Otherwise,* Γ *is small.*

We recall that $\triangle_p$ denotes the Laplace-Beltrami operator on the universal cover M.

PROPOSITION 11. *Let* X *and* M *be as above. The group* $\pi_1(X)$ *is small if and only if* $0 \in \sigma(\triangle_0)$ *or* $0 \in \sigma(\triangle_1)$.

Proof. This follows immediately from Properties 1, 2, 4, 5 and 6 above. □

The question arises as to which groups are big and which are small. Clearly any amenable group is small.

PROPOSITION 12. *Fundamental groups of compact surfaces are small.*

Proof. Suppose that Σ is a compact surface and $\Gamma = \pi_1(\Sigma)$. If Σ has boundary then Γ is a free group F_j on some number j of generators. If $j = 0$ or $j = 1$ then Γ is amenable. If $j > 1$ then $b_1^{(2)}(\Gamma) = j - 1 > 0$.

Suppose now that Σ is closed. If $\chi(\Sigma) \geq 0$ then Γ is amenable. If $\chi(\Sigma) < 0$ then $b_1^{(2)}(\Gamma) = -\chi(\Sigma) > 0$. □

We now extend Proposition 12 to 3-manifold groups. We use some facts about compact connected 3-manifolds Y, possibly with boundary. (See, for example, [21, Section 6]). Again, all of our manifolds are assumed to be oriented. First, Y has a decomposition as a connected sum $Y = Y_1 \# Y_2 \# \ldots \# Y_r$ of *prime* 3-manifolds. A prime 3-manifold is *exceptional* if it is closed and no finite cover of it is homotopy-equivalent to a Seifert, Haken or hyperbolic 3-manifold. No exceptional prime 3-manifolds are known and it is likely that there are none.

PROPOSITION 13 (Lott-Lück). *Suppose that Y is a compact connected oriented 3-manifold, possibly with boundary, none of whose prime factors are exceptional. Then $\pi_1(Y)$ is small.*

Proof. We argue by contradiction. Suppose that $\pi_1(Y)$ is big. First, $\pi_1(Y)$ must be infinite. If ∂Y has any connected components which are 2-spheres then we can cap them off with 3-balls without changing $\pi_1(Y)$. So we can assume that ∂Y does not have any 2-sphere components. In particular, $\chi(Y) = \frac{1}{2}\chi(\partial Y) \leq 0$. From [21, Theorem 0.1.1],

$$b_1^{(2)}(Y) = (r-1) - \sum_{i=1}^{r} \frac{1}{|\pi_1(Y_i)|} - \chi(Y). \tag{5.3}$$

As this must vanish, we have $\chi(Y) = 0$ and either

1. $\{|\pi_1(Y_i)|\}_{i=1}^r = \{2, 2, 1, \ldots, 1\}$ or
2. $\{|\pi_1(Y_i)|\}_{i=1}^r = \{\infty, 1, \ldots, 1\}$.

It follows that ∂Y is empty or a disjoint union of 2-tori. As there are no 2-spheres in ∂Y, if $|\pi_1(Y_i)| = 1$ then Y_i is a homotopy 3-sphere. Thus Y is homotopy-equivalent to either

1. $\mathbf{R}P^3 \# \mathbf{R}P^3$ or
2. A prime 3-manifold Y' with infinite fundamental group whose boundary is empty or a disjoint union of 2-tori.

If Y is homotopy-equivalent to $\mathbf{R}P^3\#\mathbf{R}P^3$ then $\pi_1(Y)$ is amenable, which is a contradiction. So we must be in the second case. Using Property 3, we may assume that $Y = Y'$. Then as Y is prime, it follows from [24, Chapter 1] that either $Y = S^1 \times D^2$ or Y has incompressible (or empty) boundary. If $Y = S^1 \times D^2$ then $\pi_1(Y)$ is amenable. If Y has incompressible (or empty) boundary then from [21, Theorem 0.1.5], $\alpha_2(Y) \leq 2$ unless Y is a closed 3-manifold with an $\mathbf{R}^3$, $\mathbf{R} \times S^2$ or *Sol* geometric structure. In the latter cases, Γ is amenable. Thus in any case, we get a contradiction. □

The next proposition gives examples of big groups.

PROPOSITION 14.

1. *A product of two nonamenable groups is big.*
2. *If Y is a closed nonpositively-curved locally symmetric space of dimension greater than three, with no Euclidean factors in $\widetilde{Y}$, then $\pi_1(Y)$ is big.*

Proof. 1. Suppose that $\Gamma = \Gamma_1 \times \Gamma_2$ with Γ_1 and Γ_2 nonamenable. Then Γ is nonamenable. Let K_1 and K_2 be presentation complexes with fundamental groups Γ_1 and Γ_2, respectively. Put $K = K_1 \times K_2$. Then $\Gamma = \pi_1(K)$. Let $\triangle_p(\widetilde{K})$, $\triangle_p(\widetilde{K_1})$ and $\triangle_p(\widetilde{K_2})$ denote the Laplace-Beltrami operator on p-cochains on $\widetilde{K}$, $\widetilde{K_1}$ and $\widetilde{K_2}$, respectively, as defined in Subsection 5.2 below. Then

$$\begin{aligned} \inf\big(\sigma\big(\triangle_1(\widetilde{K})\big)\big) = \min\Big(&\inf\big(\sigma\big(\triangle_1(\widetilde{K_1})\big)\big) + \inf\big(\sigma\big(\triangle_0(\widetilde{K_2})\big)\big), \\ &\inf\big(\sigma\big(\triangle_0(\widetilde{K_1})\big)\big) + \inf\big(\sigma\big(\triangle_1(\widetilde{K_2})\big)\big)\Big) > 0. \end{aligned} \tag{5.4}$$

Using Proposition 11, the first part of the proposition follows.

2. If $\widetilde{Y}$ is irreducible then part 2. of the proposition follows from the second remark after Proposition 7. If $\widetilde{Y}$ is reducible then we can use an argument similar to (5.4). □

REMARK. Let Γ be an infinite finitely-presented discrete group with Kazhdan's property T. From [6, p. 47], $\mathrm{H}^1\big(\Gamma; l^2(\Gamma)\big) = 0$. This implies that Γ is nonamenable and $b_1^{(2)}(\Gamma) = 0$. We do not know if it is necessarily true that $\alpha_2(\Gamma) = \infty^+$.

5.2 Two and Three Dimensions

In this subsection we relate the zero-in-the-spectrum question to a question in combinatorial group theory. Let K be a finite connected 2-dimensional

CW-complex. Let $\widetilde{K}$ be its universal cover. Let $C^*(\widetilde{K})$ denote the Hilbert space of square-integrable cellular cochains on $\widetilde{K}$. There is a cochain complex

$$0 \longrightarrow C^0(\widetilde{K}) \xrightarrow{d_0} C^1(\widetilde{K}) \xrightarrow{d_1} C^2(\widetilde{K}) \longrightarrow 0. \tag{5.5}$$

Define the Laplace-Beltrami operators by $\triangle_0 = d_0^* d_0$, $\triangle_1 = d_0 d_0^* + d_1^* d_1$ and $\triangle_2 = d_1 d_1^*$. These are bounded self-adjoint operators and so we can talk about zero being in the spectrum of $\widetilde{K}$.

PROPOSITION 15. *Zero is not in the spectrum of $\widetilde{K}$ if and only if $\pi_1(K)$ is big and $\chi(K) = 0$.*

Proof. Suppose that zero is not in the spectrum of $\widetilde{K}$. From the analog of Proposition 11, Γ must be big. Furthermore, from Properties 1 and 7, $\chi(K) = 0$.

Now suppose that $\pi_1(K)$ is big and $\chi(K) = 0$. From the analog of Proposition 11, $0 \notin \sigma(\triangle_0)$ and $0 \notin \sigma(\triangle_1)$. In particular, $\mathrm{Ker}(\triangle_0) = \mathrm{Ker}(\triangle_1) = 0$. From Properties 1 and 7, $\mathrm{Ker}(\triangle_2) = 0$. As $C^2(\widetilde{K}) = \mathrm{Ker}(\triangle_2) \oplus \overline{d_1 C^1(\widetilde{K})}$, we conclude that $0 \notin \sigma(\triangle_2)$. $\square$

Let Γ be a finitely-presented group. Consider a fixed presentation of Γ consisting of g generators and r relations. Let K be the corresponding presentation complex. Then $\chi(K) = 1 - g + r$. Thus zero is not in the spectrum of $\widetilde{K}$ if and only if $\pi_1(K)$ is big and $g - r = 1$.

Recall that the *deficiency* $\mathrm{def}(\Gamma)$ is defined to be the maximum, over all finite presentations of Γ, of $g - r$. If $b_1^{(2)}(\Gamma) = 0$ then from the equation

$$\chi(K) = 1 - g + r = b_0^{(2)}(\Gamma) - b_1^{(2)}(\Gamma) + b_2^{(2)}(K), \tag{5.6}$$

we obtain $\mathrm{def}(\Gamma) \leq 1$. This is the case, for example, when Γ is big or when Γ is amenable [5].

As any finite connected 2-dimensional CW-complex is homotopy-equivalent to a presentation complex, it follows from Proposition 15 that the answer to the zero-in-the-spectrum question is "yes" for universal covers of such complexes if and only if the following conjecture is true.

CONJECTURE 1. *If Γ is a big group then* $\mathrm{def}(\Gamma) \leq 0$.

REMARK. If $\pi_1(K)$ has property T then the ordinary first Betti number of K vanishes [6], and so $\chi(K) = 1 + b_2(K) > 0$. Thus zero lies in the spectrum of $\widetilde{K}$.

Now let Y be a 3-manifold satisfying the conditions of Proposition 13. If $\partial Y \neq \varnothing$, we define $\triangle_p$ on $\widetilde{Y}$ using absolute boundary conditions on $\partial\widetilde{Y}$.

PROPOSITION 16. *Zero lies in the spectrum of* $\widetilde{Y}$.

Proof. This is a consequence of Propositions 11 and 13. □

5.3 FOUR DIMENSIONS

In this subsection we relate the zero-in-the-spectrum question to a question about Euler characteristics of closed 4-dimensional manifolds.

If M is a Riemannian 4-manifold then the Hodge decomposition gives

$$\begin{aligned}
\Lambda^0(M) &= \operatorname{Ker}(\triangle_0) \oplus \Lambda^0(M)/\operatorname{Ker}(d),\\
\Lambda^1(M) &= \operatorname{Ker}(\triangle_1) \oplus \overline{d\Lambda^0(M)} \oplus \Lambda^1(M)/\operatorname{Ker}(d),\\
\Lambda^2(M) &= \operatorname{Ker}(\triangle_2) \oplus \overline{d\Lambda^1(M)} \oplus \overline{*d\Lambda^1(M)},\\
\Lambda^3(M) &= *\operatorname{Ker}(\triangle_1) \oplus \overline{*d\Lambda^0(M)} \oplus *\big(\Lambda^1(M)/\operatorname{Ker}(d)\big),\\
\Lambda^4(M) &= *\operatorname{Ker}(\triangle_0) \oplus *\big(\Lambda^0(M)/\operatorname{Ker}(d)\big).
\end{aligned} \tag{5.7}$$

Thus for the zero-in-the-spectrum question, it is enough to consider $\operatorname{Ker}(\triangle_0)$, $\operatorname{Ker}(\triangle_1)$, $\sigma\big(\triangle_0 \text{ on } \Lambda^0/\operatorname{Ker}(d)\big)$, $\sigma\big(\triangle_1 \text{ on } \Lambda^1/\operatorname{Ker}(d)\big)$ and $\operatorname{Ker}(\triangle_2)$.

Let Γ be a finitely-presented group. Recall that Γ is the fundamental group of some closed 4-manifold. To see this, take a finite presentation of Γ. Embed the resulting presentation complex in $\mathbf{R}^5$ and take the boundary of a regular neighborhood to get the manifold.

Now consider the Euler characteristics of all closed 4-manifolds X with fundamental group Γ. Given X, we have $\chi(X\#\mathbf{C}P^2) = \chi(X) + 1$. Thus it is easy to make the Euler characteristic big. However, it is not so easy to make it small. From what has been said,

$$\begin{aligned}
&\{\chi(X) \colon X \text{ is a closed connected oriented 4-manifold with}\\
&\qquad \pi_1(X) = \Gamma\} = \{n \in \mathbf{Z} : n \geq q(\Gamma)\}
\end{aligned} \tag{5.8}$$

for some $q(\Gamma)$. *A priori* $q(\Gamma) \in \mathbf{Z} \cup \{-\infty\}$, but in fact $q(\Gamma) \in \mathbf{Z}$ [17, Theorem 1]. (This also follows from (5.9) below.) It is a basic problem in 4-manifold topology to get good estimates of $q(\Gamma)$.

Suppose that $\pi_1(X) = \Gamma$. From Properties 4, 7 and 8 above,

$$\chi(X) = 2b_0^{(2)}(\Gamma) - 2b_1^{(2)}(\Gamma) + b_2^{(2)}(X)\,. \tag{5.9}$$

In particular, if $b_1^{(2)}(\Gamma) = 0$ then $\chi(X) \geq 0$ and so $q(\Gamma) \geq 0$. This is the case, for example, when Γ is big or when Γ is amenable [5].

PROPOSITION 17. *Let X be a closed 4-manifold. Then zero is not in the spectrum of $\widetilde{X}$ if and only if $\pi_1(X)$ is big and $\chi(X) = 0$.*

Proof. Suppose that zero is not in the spectrum of $\widetilde{X}$. Then from Proposition 11, $\pi_1(X)$ must be big. Furthermore, $\mathrm{Ker}(\triangle_2) = 0$. From Property 1 and (5.9), $\chi(X) = 0$.

Now suppose that $\pi_1(X)$ is big and $\chi(X) = 0$. From Proposition 11, $0 \notin \sigma(\triangle_0)$ and $0 \notin \sigma(\triangle_1)$. From Property 1 and (5.9), $\mathrm{Ker}(\triangle_2) = 0$. Then from (5.7), zero is not in the spectrum of $\widetilde{X}$. □

REMARK. If zero is not in the spectrum of $\widetilde{X}$ then it follows from Property 9 that in addition, $\tau(X) = 0$. Also, as will be explained later in Corollary 4, if $\pi_1(X)$ satisfies the Strong Novikov Conjecture then $\nu_*([X])$ vanishes in $\mathrm{H}_4(B\pi_1(X); \mathbf{C})$.

In summary, we have shown that the answer to the zero-in-the-spectrum question is "yes" for universal covers of closed 4-manifolds if and only if the following conjecture is true.

CONJECTURE 2. *If Γ is a big group then $q(\Gamma) > 0$.*

We now give some partial positive results on the zero-in-the-spectrum question for universal covers of closed 4-manifolds. Recall that there is a notion, due to Thurston, of a manifold having a geometric structure. This is especially important for 3-manifolds. The 4-manifolds with geometric structures have been studied by Wall [32].

PROPOSITION 18. *Let X be a closed 4-manifold. Then zero is in the spectrum of $\widetilde{X}$ if*

1. *$\pi_1(X)$ has property T or*
2. *X has a geometric structure (and an arbitrary Riemannian metric) or*
3. *X has a complex structure (and an arbitrary Riemannian metric).*

Proof.

1. If X has property T then the ordinary first Betti number of X vanishes [6]. Thus $\chi(X) = 2 + b_2(X) > 0$. Part 1. of the proposition follows.
2. The geometries of [32] all fall into at least one of the following classes:

a. $b_0^{(2)} \neq 0$: S^4, $S^2 \times S^2$, $\mathbf{C}P^2$.

b. $0 \in \sigma\big(\triangle_0$ on $\Lambda^0/\operatorname{Ker}(d)\big)$: $\mathbf{R}^4$, $S^3 \times \mathbf{R}$, $S^2 \times \mathbf{R}^2$, $Nil^3 \times \mathbf{R}$, Nil^4, Sol_0^4, Sol_1^4, $Sol_{m,n}^4$.

c. $b_1^{(2)} \neq 0$: $S^2 \times H^2$.

d. $0 \in \sigma\big(\triangle_1$ on $\Lambda^1/\operatorname{Ker}(d)\big)$: $H^3 \times \mathbf{R}$, $\widetilde{SL_2} \times \mathbf{R}$, $H^2 \times \mathbf{R}^2$.

e. $\chi > 0$: H^4, $H^2 \times H^2$, $\mathbf{C}H^2$.

Part 2. of the proposition follows.

3. Suppose that zero is not in the spectrum of $\widetilde{X}$. From Properties 7 and 9, $\chi(X) = \tau(X) = 0$. From the classification of complex surfaces, X has a geometric structure [32, p. 148–149]. This contradicts part 2. of the proposition. □

5.4 More Dimensions

In this subsection we give some partial positive results about the zero-in-the-spectrum question for covers of compact manifolds of arbitrary dimension. Let us first recall some facts about index theory [18]. Let X be a closed Riemannian manifold. If $\dim(X)$ is even, consider the operator $d + d^*$ on $\Lambda^*(X)$. Give $\Lambda^*(X)$ the $\mathbf{Z}_2$-grading coming from (3.12). Then the signature $\tau(X)$ equals the index of $d+d^*$. To say this in a more complicated way, the operator $d+d^*$ defines a element $[d+d^*]$ of the K-homology group $K_0(X)$. Let $\eta : X \to \mathrm{pt.}$ be the (only) map from X to a point. Then $\eta_*([d+d^*]) \in K_0(\mathrm{pt.})$. There is a map $A : K_0(\mathrm{pt.}) \to K_0(\mathbf{C})$ which is the identity, as both sides are $\mathbf{Z}$. So we can say that $\tau(X) = A\big(\eta\big([d+d^*]\big)\big) \in K_0(\mathbf{C})$.

We now extend the preceding remarks to the case of a group action. Let M be a normal cover of X with covering group Γ. The fiber bundle $M \to X$ is classified by a map $\nu : X \to B\Gamma$, defined up to homotopy. Let $\widetilde{d}$ be exterior differentiation on M. Consider the operator $\widetilde{d} + \widetilde{d}^*$. Taking into account the action of Γ on M, one can define a refined index $\operatorname{ind}(\widetilde{d} + \widetilde{d}^*) \in K_0(C_r^*\Gamma)$, where $C_r^*\Gamma$ is the reduced group C^*-algebra of Γ.

We recall the statement of the Strong Novikov Conjecture (SNC) [18, 29]. This is a conjecture about a countable discrete group Γ, namely that the assembly map $A : K_*(B\Gamma) \to K_*(C_r^*\Gamma)$ is rationally injective. Many groups of a geometric origin, such as discrete subgroups of connected Lie groups or Gromov-hyperbolic groups, are known to satisfy SNC. There are no known groups which do not satisfy SNC.

PROPOSITION 19. *Let X be a closed Riemannian manifold with a surjective homomorphism $\pi_1(X) \longrightarrow \Gamma$. Let M be the induced normal Γ-cover of X. Suppose that Γ satisfies SNC. Let $L(X) \in \mathrm{H}^*(X;\mathbf{C})$ be the Hirzebruch L-class of X and let $*L(X) \in \mathrm{H}_*(X;\mathbf{C})$ be its Poincaré dual. Then if $\nu_*(*L(X)) \neq 0$ in $H_*(B\Gamma;\mathbf{C})$, zero lies in the spectrum of M. In fact, $0 \in \sigma\left(\triangle_{\frac{\dim(X)}{2}}\right)$ if $\dim(X)$ is even and $0 \in \sigma\left(\triangle_{\frac{\dim(X)\pm 1}{2}}\right)$ if $\dim(X)$ is odd.*

Proof. Suppose first that $\dim(X)$ is even. Suppose that zero does not lie in the spectrum of M. Then the operator $\widetilde{d}+\widetilde{d}^*$ is invertible. (More precisely, it is invertible as an operator on a Hilbert $C_r^*\Gamma$-module of differential forms on M.) This implies that $\mathrm{ind}(\widetilde{d}+\widetilde{d}^*)$ vanishes in $K_0(C_r^*\Gamma)$.

The higher index theorem says that

$$\mathrm{ind}(\widetilde{d}+\widetilde{d}^*) = A\left(\nu_*\left([d+d^*]\right)\right). \tag{5.10}$$

Let $A_{\mathbf{C}} : K_0(B\Gamma) \otimes \mathbf{C} \longrightarrow K_0(C_r^*\Gamma) \otimes \mathbf{C}$ be the complexified assembly map. Using the isomorphism $K_0(B\Gamma) \otimes \mathbf{C} \cong \mathrm{H}_{even}(B\Gamma;\mathbf{C})$, the higher index theorem implies that in $K_0(C_r^*\Gamma) \otimes \mathbf{C}$,

$$\mathrm{ind}(\widetilde{d}+\widetilde{d}^*)_{\mathbf{C}} = A_{\mathbf{C}}\left(\nu_*\left(*L(X)\right)\right). \tag{5.11}$$

By assumption, $A_{\mathbf{C}}$ is injective. This gives a contradiction.

Let T be the operator obtained by restricting $\widetilde{d}+\widetilde{d}^*$ to

$$\Lambda^{\frac{\dim(X)}{2}}(M) \oplus \overline{\widetilde{d}\Lambda^{\frac{\dim(X)}{2}}(M)} \oplus \overline{*\widetilde{d}\Lambda^{\frac{\dim(X)}{2}}(M)}.$$

One can show that the other differential forms on M cancel out when computing the rational index of $\widetilde{d}+\widetilde{d}^*$, so T will have the same index as $\widetilde{d}+\widetilde{d}^*$. Then the same arguments apply to T to give $0 \in \sigma\left(\triangle_{\frac{\dim(X)}{2}}\right)$.

If $\dim(X)$ is odd, consider the even-dimensional manifold $X' = X \times S^1$ and the group $\Gamma' = \Gamma \times \mathbf{Z}$. As the proposition holds for X', it must also hold for X. □

COROLLARY 4. *Let X be a closed Riemannian manifold. Let $[X] \in \mathrm{H}_{\dim(X)}(X;\mathbf{C})$ be its fundamental class. Suppose that there is a surjective homomorphism $\pi_1(X) \longrightarrow \Gamma$ such that Γ satisfies SNC and the composite map $X \longrightarrow B\pi_1(X) \longrightarrow B\Gamma$ sends $[X]$ to a nonzero element of $\mathrm{H}_{\dim(X)}(B\Gamma;\mathbf{C})$. Let M be the induced normal Γ-cover of X. Then on M, $0 \in \sigma\left(\triangle_{\frac{\dim(X)}{2}}\right)$ if $\dim(X)$ is even and $0 \in \sigma\left(\triangle_{\frac{\dim(X)\pm 1}{2}}\right)$ if $\dim(X)$ is odd.*

Proof. As the Hirzebruch L-class starts out as $L(X) = 1 + \dots$, its Poincaré dual is of the form $*L(X) = \dots + [X]$. The corollary follows from Proposition 19. $\square$

COROLLARY 5. *Let X be a closed aspherical Riemannian manifold whose fundamental group satisfies SNC. Then on $\widetilde{X}$, $0 \in \sigma\left(\triangle_{\frac{dim(X)}{2}}\right)$ if $\dim(X)$ is even and $0 \in \sigma\left(\triangle_{\frac{dim(X)\pm 1}{2}}\right)$ if $\dim(X)$ is odd.*

Proof. This follows from Corollary 4. $\square$

EXAMPLES.

1. If $X = T^n$ then Corollary 5 is consistent with Example 2 of Section 2.
2. If X is a compact quotient of H^{2n} then Corollary 5 is consistent with Example 3 of Section 2.
3. If X is a compact quotient of H^{2n+1} then Corollary 5 is consistent with Example 4 of Section 2.
4. If X is a closed nonpositively-curved locally symmetric space then Corollary 5 is consistent with the second remark after Proposition 7.

If X is a closed aspherical manifold, it is known that SNC implies that the rational Pontryagin classes of X are homotopy-invariants [18] and that X does not admit a Riemannian metric of positive scalar curvature [29]. Thus we see that these three questions about aspherical manifolds, namely homotopy-invariance of rational Pontryagin classes, (non)existence of positive-scalar-curvature metrics and the zero-in-the-spectrum question, are roughly all on the same footing.

If X is a closed aspherical Riemannian manifold, one can ask for which p one has $0 \in \sigma(\triangle_p)$ on $\widetilde{X}$. The case of locally symmetric spaces is covered by the second remark after Proposition 7. Another interesting class of aspherical manifolds consists of those with amenable fundamental group. By [5], $\mathrm{Ker}(\triangle_p) = 0$ for all p. By Corollary 3, $0 \in \sigma(\triangle_0)$.

PROPOSITION 20. *If X is a closed aspherical manifold such that $\pi_1(X)$ has a nilpotent subgroup of finite index then $0 \in \sigma(\triangle_p)$ on $\widetilde{X}$ for all $p \in [0, \dim(X)]$.*

Proof. First, X is homotopy-equivalent to an infranilmanifold, that is, a quotient of the form $\Gamma \backslash G/K$ where K is a finite group, G is the semidirect product of K and a connected simply-connected nilpotent Lie group and Γ is a discrete cocompact subgroup of G [12, Theorem 6.4]. We may as well assume that $X = \Gamma\backslash G/K$. By passing to a finite cover, we may assume that K is trivial. That is, X is a nilmanifold. From [27, Corollary 7.28], $\mathrm{H}^p(X; \mathbf{C}) \cong \mathrm{H}^p(g, \mathbf{C})$, the Lie algebra cohomology of g. From [7], $\mathrm{H}^p(g, \mathbf{C}) \neq 0$ for all $p \in [0, \dim(X)]$. Thus for all $p \in [0, \dim(X)]$, $\mathrm{H}^p(X; \mathbf{C}) \neq 0$.

Now let ω be a nonzero harmonic p-form on X. Let $\pi^*\omega$ be its pullback to $\widetilde{X}$. The idea is to construct low-energy square-integrable p-forms on X by multiplying $\pi^*\omega$ by appropriate functions on X. We define the functions as in [2, §2]. Take a smooth triangulation of X and choose a fundamental domain F for the lifted triangulation of $\widetilde{X}$. If E is a finite subset of $\pi_1(X)$, let χ_H be the characteristic function of $H = \bigcup_{g \in E} g \cdot F$. Given numbers $0 < \epsilon_1 < \epsilon_2 < 1$, choose a nonincreasing function $\psi \in C_0^\infty\big([0,\infty)\big)$ which is identically one on $[0, \epsilon_1]$ and identically zero on $[\epsilon_2, \infty)$. Define a compactly-supported function f_E on $\widetilde{X}$ by $f_E(m) = \psi\big(d(m, H)\big)$. Then there is a constant $C_1 > 0$, independent of E, such that

$$\int_{\widetilde{X}} |df_E|^2 \leq C_1 \,\mathrm{area}(\partial H). \tag{5.12}$$

Define $\rho_E \in \Lambda^p(\widetilde{X})$ by $\rho_E = f_E \cdot \pi^*\omega$. We have $d\rho_E = df_E \wedge \pi^*\omega$ and $d^*\rho_E = -i(df_E)\,\pi^*\omega$. As f_E is identically one on H, it follows that there is a constant $C > 0$, independent of E, such that

$$\frac{\int_{\widetilde{X}} \left[|d\rho_E|^2 + |d^*\rho_E|^2\right]}{\int_{\widetilde{X}} |\rho_E|^2} \leq C \frac{\mathrm{area}(\partial H)}{\mathrm{vol}(H)}. \tag{5.13}$$

As $\pi_1(X)$ is amenable, by an appropriate choice of E this ratio can be made arbitrarily small. Thus $0 \in \sigma(\triangle_p)$. □

QUESTION. Does the conclusion of Proposition 20 hold if we only assume that $\pi_1(X)$ is amenable?

6. TOPOLOGICALLY TAME MANIFOLDS

Another class of manifolds for which one can hope to get some nontrivial results about the zero-in-the-spectrum question is given by *topologically tame* manifolds, meaning manifolds M which are diffeomorphic to the interior of a compact manifold N with boundary. If M has finite volume then $\mathrm{Ker}(\triangle_0) \neq 0$,