

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 42 (1996)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE ZERO-IN-THE-SPECTRUM QUESTION
Autor: LOTT, John
Kapitel: 5. Universal Covers
DOI: <https://doi.org/10.5169/seals-87882>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 07.03.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

5. UNIVERSAL COVERS

Suppose that M is the universal cover of a compact Riemannian manifold X . We give M the pulled-back Riemannian metric and consider the Laplace-Beltrami operator Δ_p on M . There are numerical invariants derived from $\{\Delta_p\}_{p \geq 0}$, the so-called L^2 -Betti numbers $\{b_p^{(2)}(X)\}_{p \geq 0}$ and Novikov-Shubin invariants $\{\alpha_{p+1}(X)\}_{p \geq 0}$. The L^2 -Betti numbers lie in $[0, \infty)$ and the Novikov-Shubin invariants lie in $[0, \infty] \cup \infty^+$. Here ∞^+ is a formal symbol which should be considered to be greater than ∞ . Roughly speaking, $b_p^{(2)}(X)$ measures the size of $\text{Ker}(\Delta_p)$ as a $\pi_1(X)$ -module and $\alpha_{p+1}(X)$ measures the thickness near zero of the spectrum of Δ_p on $\Lambda^p(M)/\text{Ker}(d)$; the larger $\alpha_{p+1}(X)$, the thinner the spectrum near zero. We refer to [21, 22, 26] for the definitions of these invariants. We will only need the following properties:

PROPERTIES.

1. $b_p^{(2)}(X) = 0 \iff \text{Ker}(\Delta_p) = 0$.
2. $0 \notin \sigma(\Delta_p \text{ on } \Lambda^p(M)/\text{Ker}(d)) \iff \alpha_{p+1} = \infty^+$.
3. $b_p^{(2)}(X)$ and $\alpha_p(X)$ are homotopy-invariants of X .
4. $b_0^{(2)}(X)$, $b_1^{(2)}(X)$, $\alpha_1(X)$ and $\alpha_2(X)$ only depend on $\pi_1(X)$.
5. $b_0^{(2)}(X) = 0$ if and only if $\pi_1(X)$ is infinite.
6. $\alpha_1(X) = \infty^+$ if and only if $\pi_1(X)$ is finite or nonamenable.
7. The Euler characteristic of X satisfies

$$(5.1) \quad \chi(X) = \sum_p (-1)^p b_p^{(2)}(X)$$

8. If X^n is closed then $b_{n-p}^{(2)}(X) = b_p^{(2)}(X)$.
9. If X^{4k} is closed then there are nonnegative numbers $b_{2k, \pm}^{(2)}(X)$ such that $b_{2k}^{(2)}(X) = b_{2k, +}^{(2)}(X) + b_{2k, -}^{(2)}(X)$ and the signature of X satisfies

$$(5.2) \quad \tau(X) = b_{2k, +}^{(2)}(X) - b_{2k, -}^{(2)}(X).$$

One can extend properties 1-7 from compact Riemannian manifolds X to finite CW-complexes K .

In what follows, Γ will denote a finitely-presented group. Given a presentation of Γ , there is an associated 2-dimensional CW-complex K which we call the *presentation complex*. To form it, make a bouquet of circles indexed by the generators of Γ . Attach 2-cells based on the relations of Γ . (We allow trivial or repeated relations in the presentation.) This is the presentation complex.

DEFINITION 7. Put $b_0^{(2)}(\Gamma) = b_0^{(2)}(K)$, $b_1^{(2)}(\Gamma) = b_1^{(2)}(K)$, $\alpha_1(\Gamma) = \alpha_1(K)$ and $\alpha_2(\Gamma) = \alpha_2(K)$.

By Property 4 above, Definition 7 makes sense in that the choice of presentation of Γ does not matter.

As the invariants $b_0^{(2)}(\Gamma)$, $b_1^{(2)}(\Gamma)$, $\alpha_1(\Gamma)$ and $\alpha_2(\Gamma)$ will play an important role, let us state explicitly what they measure. First, from Property 5, $b_0^{(2)}(\Gamma)$ tells us whether or not Γ is infinite. In general, $b_0^{(2)}(\Gamma) = \frac{1}{|\Gamma|}$. Next, from Property 1, $b_1^{(2)}(\Gamma)$ tells us whether or not M has square-integrable harmonic 1-forms (or \tilde{K} has square-integrable harmonic 1-cochains). From Property 2, $\alpha_1(\Gamma)$ tells us whether or not the Laplacian Δ_0 , acting on functions on M , has a gap in its spectrum away from zero. In fact, Property 6 is just a restatement of Corollary 3. Finally, from Property 2, $\alpha_2(\Gamma)$ tells us whether or not the spectrum of the Laplacian on $\Lambda^1(M)/\text{Ker}(d)$ goes down to zero.

5.1 BIG AND SMALL GROUPS

Let us first introduce a convenient terminology for the purposes of the present paper.

DEFINITION 8. *The group Γ is big if it is nonamenable, $b_1^{(2)}(\Gamma) = 0$ and $\alpha_2(\Gamma) = \infty^+$. Otherwise, Γ is small.*

We recall that Δ_p denotes the Laplace-Beltrami operator on the universal cover M .

PROPOSITION 11. *Let X and M be as above. The group $\pi_1(X)$ is small if and only if $0 \in \sigma(\Delta_0)$ or $0 \in \sigma(\Delta_1)$.*

Proof. This follows immediately from Properties 1, 2, 4, 5 and 6 above. \square

The question arises as to which groups are big and which are small. Clearly any amenable group is small.

PROPOSITION 12. *Fundamental groups of compact surfaces are small.*

Proof. Suppose that Σ is a compact surface and $\Gamma = \pi_1(\Sigma)$. If Σ has boundary then Γ is a free group F_j on some number j of generators. If $j = 0$ or $j = 1$ then Γ is amenable. If $j > 1$ then $b_1^{(2)}(\Gamma) = j - 1 > 0$.

Suppose now that Σ is closed. If $\chi(\Sigma) \geq 0$ then Γ is amenable. If $\chi(\Sigma) < 0$ then $b_1^{(2)}(\Gamma) = -\chi(\Sigma) > 0$. \square

We now extend Proposition 12 to 3-manifold groups. We use some facts about compact connected 3-manifolds Y , possibly with boundary. (See, for example, [21, Section 6]). Again, all of our manifolds are assumed to be oriented. First, Y has a decomposition as a connected sum $Y = Y_1 \# Y_2 \# \dots \# Y_r$ of *prime* 3-manifolds. A prime 3-manifold is *exceptional* if it is closed and no finite cover of it is homotopy-equivalent to a Seifert, Haken or hyperbolic 3-manifold. No exceptional prime 3-manifolds are known and it is likely that there are none.

PROPOSITION 13 (Lott-Lück). *Suppose that Y is a compact connected oriented 3-manifold, possibly with boundary, none of whose prime factors are exceptional. Then $\pi_1(Y)$ is small.*

Proof. We argue by contradiction. Suppose that $\pi_1(Y)$ is big. First, $\pi_1(Y)$ must be infinite. If ∂Y has any connected components which are 2-spheres then we can cap them off with 3-balls without changing $\pi_1(Y)$. So we can assume that ∂Y does not have any 2-sphere components. In particular, $\chi(Y) = \frac{1}{2}\chi(\partial Y) \leq 0$. From [21, Theorem 0.1.1],

$$(5.3) \quad b_1^{(2)}(Y) = (r - 1) - \sum_{i=1}^r \frac{1}{|\pi_1(Y_i)|} - \chi(Y).$$

As this must vanish, we have $\chi(Y) = 0$ and either

1. $\{|\pi_1(Y_i)|\}_{i=1}^r = \{2, 2, 1, \dots, 1\}$ or
2. $\{|\pi_1(Y_i)|\}_{i=1}^r = \{\infty, 1, \dots, 1\}$.

It follows that ∂Y is empty or a disjoint union of 2-tori. As there are no 2-spheres in ∂Y , if $|\pi_1(Y_i)| = 1$ then Y_i is a homotopy 3-sphere. Thus Y is homotopy-equivalent to either

1. $\mathbf{RP}^3 \# \mathbf{RP}^3$ or
2. A prime 3-manifold Y' with infinite fundamental group whose boundary is empty or a disjoint union of 2-tori.

If Y is homotopy-equivalent to $\mathbf{R}P^3\#\mathbf{R}P^3$ then $\pi_1(Y)$ is amenable, which is a contradiction. So we must be in the second case. Using Property 3, we may assume that $Y = Y'$. Then as Y is prime, it follows from [24, Chapter 1] that either $Y = S^1 \times D^2$ or Y has incompressible (or empty) boundary. If $Y = S^1 \times D^2$ then $\pi_1(Y)$ is amenable. If Y has incompressible (or empty) boundary then from [21, Theorem 0.1.5], $\alpha_2(Y) \leq 2$ unless Y is a closed 3-manifold with an \mathbf{R}^3 , $\mathbf{R} \times S^2$ or Sol geometric structure. In the latter cases, Γ is amenable. Thus in any case, we get a contradiction. \square

The next proposition gives examples of big groups.

PROPOSITION 14.

1. *A product of two nonamenable groups is big.*
2. *If Y is a closed nonpositively-curved locally symmetric space of dimension greater than three, with no Euclidean factors in \tilde{Y} , then $\pi_1(Y)$ is big.*

Proof. 1. Suppose that $\Gamma = \Gamma_1 \times \Gamma_2$ with Γ_1 and Γ_2 nonamenable. Then Γ is nonamenable. Let K_1 and K_2 be presentation complexes with fundamental groups Γ_1 and Γ_2 , respectively. Put $K = K_1 \times K_2$. Then $\Gamma = \pi_1(K)$. Let $\Delta_p(\tilde{K})$, $\Delta_p(\tilde{K}_1)$ and $\Delta_p(\tilde{K}_2)$ denote the Laplace-Beltrami operator on p -cochains on \tilde{K} , \tilde{K}_1 and \tilde{K}_2 , respectively, as defined in Subsection 5.2 below. Then

$$(5.4) \quad \inf(\sigma(\Delta_1(\tilde{K}))) = \min(\inf(\sigma(\Delta_1(\tilde{K}_1))) + \inf(\sigma(\Delta_0(\tilde{K}_2))), \inf(\sigma(\Delta_0(\tilde{K}_1))) + \inf(\sigma(\Delta_1(\tilde{K}_2)))) > 0.$$

Using Proposition 11, the first part of the proposition follows.

2. If \tilde{Y} is irreducible then part 2. of the proposition follows from the second remark after Proposition 7. If \tilde{Y} is reducible then we can use an argument similar to (5.4). \square

REMARK. Let Γ be an infinite finitely-presented discrete group with Kazhdan's property T. From [6, p. 47], $H^1(\Gamma; \ell^2(\Gamma)) = 0$. This implies that Γ is nonamenable and $b_1^{(2)}(\Gamma) = 0$. We do not know if it is necessarily true that $\alpha_2(\Gamma) = \infty^+$.

5.2 TWO AND THREE DIMENSIONS

In this subsection we relate the zero-in-the-spectrum question to a question in combinatorial group theory. Let K be a finite connected 2-dimensional

CW-complex. Let \tilde{K} be its universal cover. Let $C^*(\tilde{K})$ denote the Hilbert space of square-integrable cellular cochains on \tilde{K} . There is a cochain complex

$$(5.5) \quad 0 \longrightarrow C^0(\tilde{K}) \xrightarrow{d_0} C^1(\tilde{K}) \xrightarrow{d_1} C^2(\tilde{K}) \longrightarrow 0.$$

Define the Laplace-Beltrami operators by $\Delta_0 = d_0^*d_0$, $\Delta_1 = d_0d_0^* + d_1^*d_1$ and $\Delta_2 = d_1d_1^*$. These are bounded self-adjoint operators and so we can talk about zero being in the spectrum of \tilde{K} .

PROPOSITION 15. *Zero is not in the spectrum of \tilde{K} if and only if $\pi_1(K)$ is big and $\chi(K) = 0$.*

Proof. Suppose that zero is not in the spectrum of \tilde{K} . From the analog of Proposition 11, Γ must be big. Furthermore, from Properties 1 and 7, $\chi(K) = 0$.

Now suppose that $\pi_1(K)$ is big and $\chi(K) = 0$. From the analog of Proposition 11, $0 \notin \sigma(\Delta_0)$ and $0 \notin \sigma(\Delta_1)$. In particular, $\text{Ker}(\Delta_0) = \text{Ker}(\Delta_1) = 0$. From Properties 1 and 7, $\text{Ker}(\Delta_2) = 0$. As $C^2(\tilde{K}) = \text{Ker}(\Delta_2) \oplus d_1C^1(\tilde{K})$, we conclude that $0 \notin \sigma(\Delta_2)$. \square

Let Γ be a finitely-presented group. Consider a fixed presentation of Γ consisting of g generators and r relations. Let K be the corresponding presentation complex. Then $\chi(K) = 1 - g + r$. Thus zero is not in the spectrum of \tilde{K} if and only if $\pi_1(K)$ is big and $g - r = 1$.

Recall that the *deficiency* $\text{def}(\Gamma)$ is defined to be the maximum, over all finite presentations of Γ , of $g - r$. If $b_1^{(2)}(\Gamma) = 0$ then from the equation

$$(5.6) \quad \chi(K) = 1 - g + r = b_0^{(2)}(\Gamma) - b_1^{(2)}(\Gamma) + b_2^{(2)}(K),$$

we obtain $\text{def}(\Gamma) \leq 1$. This is the case, for example, when Γ is big or when Γ is amenable [5].

As any finite connected 2-dimensional CW-complex is homotopy-equivalent to a presentation complex, it follows from Proposition 15 that the answer to the zero-in-the-spectrum question is “yes” for universal covers of such complexes if and only if the following conjecture is true.

CONJECTURE 1. *If Γ is a big group then $\text{def}(\Gamma) \leq 0$.*

REMARK. If $\pi_1(K)$ has property T then the ordinary first Betti number of K vanishes [6], and so $\chi(K) = 1 + b_2(K) > 0$. Thus zero lies in the spectrum of \tilde{K} .

Now let Y be a 3-manifold satisfying the conditions of Proposition 13. If $\partial Y \neq \emptyset$, we define Δ_p on \tilde{Y} using absolute boundary conditions on $\partial\tilde{Y}$.

PROPOSITION 16. *Zero lies in the spectrum of \tilde{Y} .*

Proof. This is a consequence of Propositions 11 and 13. \square

5.3 FOUR DIMENSIONS

In this subsection we relate the zero-in-the-spectrum question to a question about Euler characteristics of closed 4-dimensional manifolds.

If M is a Riemannian 4-manifold then the Hodge decomposition gives

$$\begin{aligned}
 (5.7) \quad \Lambda^0(M) &= \text{Ker}(\Delta_0) \oplus \Lambda^0(M) / \text{Ker}(d), \\
 \Lambda^1(M) &= \text{Ker}(\Delta_1) \oplus \overline{d\Lambda^0(M)} \oplus \Lambda^1(M) / \text{Ker}(d), \\
 \Lambda^2(M) &= \text{Ker}(\Delta_2) \oplus \overline{d\Lambda^1(M)} \oplus *d\Lambda^1(M), \\
 \Lambda^3(M) &= *\text{Ker}(\Delta_1) \oplus *\overline{d\Lambda^0(M)} \oplus *(\Lambda^1(M) / \text{Ker}(d)), \\
 \Lambda^4(M) &= *\text{Ker}(\Delta_0) \oplus *(\Lambda^0(M) / \text{Ker}(d)).
 \end{aligned}$$

Thus for the zero-in-the-spectrum question, it is enough to consider $\text{Ker}(\Delta_0)$, $\text{Ker}(\Delta_1)$, $\sigma(\Delta_0 \text{ on } \Lambda^0 / \text{Ker}(d))$, $\sigma(\Delta_1 \text{ on } \Lambda^1 / \text{Ker}(d))$ and $\text{Ker}(\Delta_2)$.

Let Γ be a finitely-presented group. Recall that Γ is the fundamental group of some closed 4-manifold. To see this, take a finite presentation of Γ . Embed the resulting presentation complex in \mathbf{R}^5 and take the boundary of a regular neighborhood to get the manifold.

Now consider the Euler characteristics of all closed 4-manifolds X with fundamental group Γ . Given X , we have $\chi(X\#\mathbf{C}P^2) = \chi(X) + 1$. Thus it is easy to make the Euler characteristic big. However, it is not so easy to make it small. From what has been said,

$$\begin{aligned}
 (5.8) \quad &\{\chi(X) : X \text{ is a closed connected oriented 4-manifold with} \\
 &\pi_1(X) = \Gamma\} = \{n \in \mathbf{Z} : n \geq q(\Gamma)\}
 \end{aligned}$$

for some $q(\Gamma)$. *A priori* $q(\Gamma) \in \mathbf{Z} \cup \{-\infty\}$, but in fact $q(\Gamma) \in \mathbf{Z}$ [17, Theorem 1]. (This also follows from (5.9) below.) It is a basic problem in 4-manifold topology to get good estimates of $q(\Gamma)$.

Suppose that $\pi_1(X) = \Gamma$. From Properties 4, 7 and 8 above,

$$(5.9) \quad \chi(X) = 2b_0^{(2)}(\Gamma) - 2b_1^{(2)}(\Gamma) + b_2^{(2)}(X).$$

In particular, if $b_1^{(2)}(\Gamma) = 0$ then $\chi(X) \geq 0$ and so $q(\Gamma) \geq 0$. This is the case, for example, when Γ is big or when Γ is amenable [5].

PROPOSITION 17. *Let X be a closed 4-manifold. Then zero is not in the spectrum of \tilde{X} if and only if $\pi_1(X)$ is big and $\chi(X) = 0$.*

Proof. Suppose that zero is not in the spectrum of \tilde{X} . Then from Proposition 11, $\pi_1(X)$ must be big. Furthermore, $\text{Ker}(\Delta_2) = 0$. From Property 1 and (5.9), $\chi(X) = 0$.

Now suppose that $\pi_1(X)$ is big and $\chi(X) = 0$. From Proposition 11, $0 \notin \sigma(\Delta_0)$ and $0 \notin \sigma(\Delta_1)$. From Property 1 and (5.9), $\text{Ker}(\Delta_2) = 0$. Then from (5.7), zero is not in the spectrum of \tilde{X} . \square

REMARK. If zero is not in the spectrum of \tilde{X} then it follows from Property 9 that in addition, $\tau(X) = 0$. Also, as will be explained later in Corollary 4, if $\pi_1(X)$ satisfies the Strong Novikov Conjecture then $\nu_*([X])$ vanishes in $H_4(B\pi_1(X); \mathbf{C})$.

In summary, we have shown that the answer to the zero-in-the-spectrum question is “yes” for universal covers of closed 4-manifolds if and only if the following conjecture is true.

CONJECTURE 2. *If Γ is a big group then $q(\Gamma) > 0$.*

We now give some partial positive results on the zero-in-the-spectrum question for universal covers of closed 4-manifolds. Recall that there is a notion, due to Thurston, of a manifold having a geometric structure. This is especially important for 3-manifolds. The 4-manifolds with geometric structures have been studied by Wall [32].

PROPOSITION 18. *Let X be a closed 4-manifold. Then zero is in the spectrum of \tilde{X} if*

1. $\pi_1(X)$ has property T or
2. X has a geometric structure (and an arbitrary Riemannian metric) or
3. X has a complex structure (and an arbitrary Riemannian metric).

Proof.

1. If X has property T then the ordinary first Betti number of X vanishes [6]. Thus $\chi(X) = 2 + b_2(X) > 0$. Part 1. of the proposition follows.
2. The geometries of [32] all fall into at least one of the following classes :

- a. $b_0^{(2)} \neq 0 : S^4, S^2 \times S^2, \mathbf{CP}^2$.
- b. $0 \in \sigma(\Delta_0 \text{ on } \Lambda^0 / \text{Ker}(d)) : \mathbf{R}^4, S^3 \times \mathbf{R}, S^2 \times \mathbf{R}^2, Nil^3 \times \mathbf{R}, Nil^4, Sol_0^4, Sol_1^4, Sol_{m,n}^4$.
- c. $b_1^{(2)} \neq 0 : S^2 \times H^2$.
- d. $0 \in \sigma(\Delta_1 \text{ on } \Lambda^1 / \text{Ker}(d)) : H^3 \times \mathbf{R}, \widetilde{SL}_2 \times \mathbf{R}, H^2 \times \mathbf{R}^2$.
- e. $\chi > 0 : H^4, H^2 \times H^2, \mathbf{CH}^2$.

Part 2. of the proposition follows.

- 3. Suppose that zero is not in the spectrum of \widetilde{X} . From Properties 7 and 9, $\chi(X) = \tau(X) = 0$. From the classification of complex surfaces, X has a geometric structure [32, p. 148–149]. This contradicts part 2. of the proposition. \square

5.4 MORE DIMENSIONS

In this subsection we give some partial positive results about the zero-in-the-spectrum question for covers of compact manifolds of arbitrary dimension. Let us first recall some facts about index theory [18]. Let X be a closed Riemannian manifold. If $\dim(X)$ is even, consider the operator $d + d^*$ on $\Lambda^*(X)$. Give $\Lambda^*(X)$ the \mathbf{Z}_2 -grading coming from (3.12). Then the signature $\tau(X)$ equals the index of $d + d^*$. To say this in a more complicated way, the operator $d + d^*$ defines a element $[d + d^*]$ of the K-homology group $K_0(X)$. Let $\eta : X \rightarrow \text{pt.}$ be the (only) map from X to a point. Then $\eta_*([d + d^*]) \in K_0(\text{pt.})$. There is a map $A : K_0(\text{pt.}) \rightarrow K_0(\mathbf{C})$ which is the identity, as both sides are \mathbf{Z} . So we can say that $\tau(X) = A(\eta_*([d + d^*])) \in K_0(\mathbf{C})$.

We now extend the preceding remarks to the case of a group action. Let M be a normal cover of X with covering group Γ . The fiber bundle $M \rightarrow X$ is classified by a map $\nu : X \rightarrow B\Gamma$, defined up to homotopy. Let \widetilde{d} be exterior differentiation on M . Consider the operator $\widetilde{d} + \widetilde{d}^*$. Taking into account the action of Γ on M , one can define a refined index $\text{ind}(\widetilde{d} + \widetilde{d}^*) \in K_0(C_r^*\Gamma)$, where $C_r^*\Gamma$ is the reduced group C^* -algebra of Γ .

We recall the statement of the Strong Novikov Conjecture (SNC) [18, 29]. This is a conjecture about a countable discrete group Γ , namely that the assembly map $A : K_*(B\Gamma) \rightarrow K_*(C_r^*\Gamma)$ is rationally injective. Many groups of a geometric origin, such as discrete subgroups of connected Lie groups or Gromov-hyperbolic groups, are known to satisfy SNC. There are no known groups which do not satisfy SNC.

PROPOSITION 19. *Let X be a closed Riemannian manifold with a surjective homomorphism $\pi_1(X) \rightarrow \Gamma$. Let M be the induced normal Γ -cover of X . Suppose that Γ satisfies SNC. Let $L(X) \in H^*(X; \mathbf{C})$ be the Hirzebruch L -class of X and let $*L(X) \in H_*(X; \mathbf{C})$ be its Poincaré dual. Then if $\nu_*(*L(X)) \neq 0$ in $H_*(B\Gamma; \mathbf{C})$, zero lies in the spectrum of M . In fact, $0 \in \sigma \left(\Delta_{\frac{\dim(X)}{2}} \right)$ if $\dim(X)$ is even and $0 \in \sigma \left(\Delta_{\frac{\dim(X) \pm 1}{2}} \right)$ if $\dim(X)$ is odd.*

Proof. Suppose first that $\dim(X)$ is even. Suppose that zero does not lie in the spectrum of M . Then the operator $\tilde{d} + \tilde{d}^*$ is invertible. (More precisely, it is invertible as an operator on a Hilbert $C_r^*\Gamma$ -module of differential forms on M .) This implies that $\text{ind}(\tilde{d} + \tilde{d}^*)$ vanishes in $K_0(C_r^*\Gamma)$.

The higher index theorem says that

$$(5.10) \quad \text{ind}(\tilde{d} + \tilde{d}^*) = A(\nu_*([d + d^*])).$$

Let $A_{\mathbf{C}} : K_0(B\Gamma) \otimes \mathbf{C} \rightarrow K_0(C_r^*\Gamma) \otimes \mathbf{C}$ be the complexified assembly map. Using the isomorphism $K_0(B\Gamma) \otimes \mathbf{C} \cong H_{\text{even}}(B\Gamma; \mathbf{C})$, the higher index theorem implies that in $K_0(C_r^*\Gamma) \otimes \mathbf{C}$,

$$(5.11) \quad \text{ind}(\tilde{d} + \tilde{d}^*)_{\mathbf{C}} = A_{\mathbf{C}}(\nu_*(*L(X))).$$

By assumption, $A_{\mathbf{C}}$ is injective. This gives a contradiction.

Let T be the operator obtained by restricting $\tilde{d} + \tilde{d}^*$ to

$$\Lambda^{\frac{\dim(X)}{2}}(M) \oplus \overline{\tilde{d}\Lambda^{\frac{\dim(X)}{2}}(M)} \oplus \overline{*d\Lambda^{\frac{\dim(X)}{2}}(M)}.$$

One can show that the other differential forms on M cancel out when computing the rational index of $\tilde{d} + \tilde{d}^*$, so T will have the same index as $\tilde{d} + \tilde{d}^*$. Then the same arguments apply to T to give $0 \in \sigma \left(\Delta_{\frac{\dim(X)}{2}} \right)$.

If $\dim(X)$ is odd, consider the even-dimensional manifold $X' = X \times S^1$ and the group $\Gamma' = \Gamma \times \mathbf{Z}$. As the proposition holds for X' , it must also hold for X . \square

COROLLARY 4. *Let X be a closed Riemannian manifold. Let $[X] \in H_{\dim(X)}(X; \mathbf{C})$ be its fundamental class. Suppose that there is a surjective homomorphism $\pi_1(X) \rightarrow \Gamma$ such that Γ satisfies SNC and the composite map $X \rightarrow B\pi_1(X) \rightarrow B\Gamma$ sends $[X]$ to a nonzero element of $H_{\dim(X)}(B\Gamma; \mathbf{C})$. Let M be the induced normal Γ -cover of X . Then on M , $0 \in \sigma \left(\Delta_{\frac{\dim(X)}{2}} \right)$ if $\dim(X)$ is even and $0 \in \sigma \left(\Delta_{\frac{\dim(X) \pm 1}{2}} \right)$ if $\dim(X)$ is odd.*

Proof. As the Hirzebruch L -class starts out as $L(X) = 1 + \dots$, its Poincaré dual is of the form $*L(X) = \dots + [X]$. The corollary follows from Proposition 19. \square

COROLLARY 5. *Let X be a closed aspherical Riemannian manifold whose fundamental group satisfies SNC. Then on \tilde{X} , $0 \in \sigma\left(\Delta_{\frac{\dim(X)}{2}}\right)$ if $\dim(X)$ is even and $0 \in \sigma\left(\Delta_{\frac{\dim(X)\pm 1}{2}}\right)$ if $\dim(X)$ is odd.*

Proof. This follows from Corollary 4. \square

EXAMPLES.

1. If $X = T^n$ then Corollary 5 is consistent with Example 2 of Section 2.
2. If X is a compact quotient of H^{2n} then Corollary 5 is consistent with Example 3 of Section 2.
3. If X is a compact quotient of H^{2n+1} then Corollary 5 is consistent with Example 4 of Section 2.
4. If X is a closed nonpositively-curved locally symmetric space then Corollary 5 is consistent with the second remark after Proposition 7.

If X is a closed aspherical manifold, it is known that SNC implies that the rational Pontryagin classes of X are homotopy-invariants [18] and that X does not admit a Riemannian metric of positive scalar curvature [29]. Thus we see that these three questions about aspherical manifolds, namely homotopy-invariance of rational Pontryagin classes, (non)existence of positive-scalar-curvature metrics and the zero-in-the-spectrum question, are roughly all on the same footing.

If X is a closed aspherical Riemannian manifold, one can ask for which p one has $0 \in \sigma(\Delta_p)$ on \tilde{X} . The case of locally symmetric spaces is covered by the second remark after Proposition 7. Another interesting class of aspherical manifolds consists of those with amenable fundamental group. By [5], $\text{Ker}(\Delta_p) = 0$ for all p . By Corollary 3, $0 \in \sigma(\Delta_0)$.

PROPOSITION 20. *If X is a closed aspherical manifold such that $\pi_1(X)$ has a nilpotent subgroup of finite index then $0 \in \sigma(\Delta_p)$ on \tilde{X} for all $p \in [0, \dim(X)]$.*

Proof. First, X is homotopy-equivalent to an infranilmanifold, that is, a quotient of the form $\Gamma \backslash G / K$ where K is a finite group, G is the

semidirect product of K and a connected simply-connected nilpotent Lie group and Γ is a discrete cocompact subgroup of G [12, Theorem 6.4]. We may as well assume that $X = \Gamma \backslash G/K$. By passing to a finite cover, we may assume that K is trivial. That is, X is a nilmanifold. From [27, Corollary 7.28], $H^p(X; \mathbf{C}) \cong H^p(\mathfrak{g}, \mathbf{C})$, the Lie algebra cohomology of \mathfrak{g} . From [7], $H^p(\mathfrak{g}, \mathbf{C}) \neq 0$ for all $p \in [0, \dim(X)]$. Thus for all $p \in [0, \dim(X)]$, $H^p(X; \mathbf{C}) \neq 0$.

Now let ω be a nonzero harmonic p -form on X . Let $\pi^*\omega$ be its pullback to \tilde{X} . The idea is to construct low-energy square-integrable p -forms on X by multiplying $\pi^*\omega$ by appropriate functions on X . We define the functions as in [2, §2]. Take a smooth triangulation of X and choose a fundamental domain F for the lifted triangulation of \tilde{X} . If E is a finite subset of $\pi_1(X)$, let χ_H be the characteristic function of $H = \bigcup_{g \in E} g \cdot F$. Given numbers $0 < \epsilon_1 < \epsilon_2 < 1$, choose a nonincreasing function $\psi \in C_0^\infty([0, \infty))$ which is identically one on $[0, \epsilon_1]$ and identically zero on $[\epsilon_2, \infty)$. Define a compactly-supported function f_E on \tilde{X} by $f_E(m) = \psi(d(m, H))$. Then there is a constant $C_1 > 0$, independent of E , such that

$$(5.12) \quad \int_{\tilde{X}} |df_E|^2 \leq C_1 \text{area}(\partial H).$$

Define $\rho_E \in \Lambda^p(\tilde{X})$ by $\rho_E = f_E \cdot \pi^*\omega$. We have $d\rho_E = df_E \wedge \pi^*\omega$ and $d^*\rho_E = -i(df_E)\pi^*\omega$. As f_E is identically one on H , it follows that there is a constant $C > 0$, independent of E , such that

$$(5.13) \quad \frac{\int_{\tilde{X}} [|d\rho_E|^2 + |d^*\rho_E|^2]}{\int_{\tilde{X}} |\rho_E|^2} \leq C \frac{\text{area}(\partial H)}{\text{vol}(H)}.$$

As $\pi_1(X)$ is amenable, by an appropriate choice of E this ratio can be made arbitrarily small. Thus $0 \in \sigma(\Delta_p)$. \square

QUESTION. Does the conclusion of Proposition 20 hold if we only assume that $\pi_1(X)$ is amenable?

6. TOPOLOGICALLY TAME MANIFOLDS

Another class of manifolds for which one can hope to get some nontrivial results about the zero-in-the-spectrum question is given by *topologically tame* manifolds, meaning manifolds M which are diffeomorphic to the interior of a compact manifold N with boundary. If M has finite volume then $\text{Ker}(\Delta_0) \neq 0$,