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4.2 Two Dimensions

Proposition 10 (Lott, Dodziuk). *The answer to the zero-in-the-spectrum question is "yes" if M is a two-dimensional manifold.*

Proof. The Hodge decomposition gives

$$\Lambda^0(M) = \operatorname{Ker}(\triangle_0) \oplus \Lambda^0(M)/\operatorname{Ker}(d), \tag{4.4}$$

$$\Lambda^1(M) = \operatorname{Ker}(\triangle_1) \oplus \overline{d\Lambda^0(M)} \oplus \overline{*d\Lambda^0(M)}, \tag{4.5}$$

$$\Lambda^2(M) = *\operatorname{Ker}(\triangle_0) \oplus *\big(\Lambda^0(M)/\operatorname{Ker}(d)\big). \tag{4.6}$$

Thus it is enough to look at

$$\operatorname{Ker}(\triangle_0), \quad \operatorname{Ker}(\triangle_1) \quad \text{and} \quad \sigma\left(\triangle_0 \text{ on } \Lambda^0(M)/\operatorname{Ker}(d)\right).$$

We argue by contradiction. Assume that zero is not in the spectrum. By Proposition 4, $\operatorname{Im}\big(\mathrm{H}^1_c(M) \to \mathrm{H}^1(M)\big) = 0$. Thus M must be planar, in the sense of either of the following two equivalent conditions:

1. Any simple closed curve in M separates it into two pieces.
2. M is diffeomorphic to the complement of a closed subset of S^2.

As $\operatorname{Ker}(\triangle_0) = 0$, M cannot be S^2. By Proposition 5, the possible existence of nonzero square-integrable harmonic 1-forms on M only depends on the underlying Riemann surface coming from the Riemannian metric on M.

We recall some notions from Riemann surface theory [1]. A function $f \in C^\infty(M)$ is *superharmonic* if $\triangle_0 f > 0$. (This is a conformally-invariant statement.) The Riemann surface underlying M is *hyperbolic* if it has a positive superharmonic function and *parabolic* otherwise. If M is planar and hyperbolic then there is a nonconstant harmonic function $f \in C^\infty(M)$ such that $\int_M df \wedge *df < \infty$ [1, p. 208]. Then df would be a nonzero element of $\operatorname{Ker}(\triangle_1)$. Thus M must be parabolic.

Put $\lambda_0 = \inf\big(\sigma(\triangle_0)\big)$. Choose some λ such that $0 < \lambda < \lambda_0$. Then there is a positive $f \in C^\infty(M)$ (not square-integrable !) such that $\triangle_0 f = \lambda f$ [31, Theorem 2.1]. However, this contradicts the parabolicity of M. □

We do not know of any result analogous to Proposition 10 for general two-dimensional simplicial complexes, say uniformly finite. See, however, Subsection 5.2.