

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 42 (1996)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE ZERO-IN-THE-SPECTRUM QUESTION
Autor: LOTT, John
Kapitel: 4.2 Two Dimensions
DOI: <https://doi.org/10.5169/seals-87882>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 13.03.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

4.2 TWO DIMENSIONS

PROPOSITION 10 (Lott, Dodziuk). *The answer to the zero-in-the-spectrum question is “yes” if M is a two-dimensional manifold.*

Proof. The Hodge decomposition gives

$$(4.4) \quad \Lambda^0(M) = \text{Ker}(\Delta_0) \oplus \Lambda^0(M) / \text{Ker}(d),$$

$$(4.5) \quad \Lambda^1(M) = \text{Ker}(\Delta_1) \oplus \overline{d\Lambda^0(M)} \oplus \overline{*d\Lambda^0(M)},$$

$$(4.6) \quad \Lambda^2(M) = *\text{Ker}(\Delta_0) \oplus *(\Lambda^0(M) / \text{Ker}(d)).$$

Thus it is enough to look at

$$\text{Ker}(\Delta_0), \quad \text{Ker}(\Delta_1) \quad \text{and} \quad \sigma(\Delta_0 \text{ on } \Lambda^0(M) / \text{Ker}(d)).$$

We argue by contradiction. Assume that zero is not in the spectrum. By Proposition 4, $\text{Im}(\mathbb{H}_c^1(M) \rightarrow \mathbb{H}^1(M)) = 0$. Thus M must be planar, in the sense of either of the following two equivalent conditions:

1. Any simple closed curve in M separates it into two pieces.
2. M is diffeomorphic to the complement of a closed subset of S^2 .

As $\text{Ker}(\Delta_0) = 0$, M cannot be S^2 . By Proposition 5, the possible existence of nonzero square-integrable harmonic 1-forms on M only depends on the underlying Riemann surface coming from the Riemannian metric on M .

We recall some notions from Riemann surface theory [1]. A function $f \in C^\infty(M)$ is *superharmonic* if $\Delta_0 f > 0$. (This is a conformally-invariant statement.) The Riemann surface underlying M is *hyperbolic* if it has a positive superharmonic function and *parabolic* otherwise. If M is planar and hyperbolic then there is a nonconstant harmonic function $f \in C^\infty(M)$ such that $\int_M df \wedge *df < \infty$ [1, p. 208]. Then df would be a nonzero element of $\text{Ker}(\Delta_1)$. Thus M must be parabolic.

Put $\lambda_0 = \inf(\sigma(\Delta_0))$. Choose some λ such that $0 < \lambda < \lambda_0$. Then there is a positive $f \in C^\infty(M)$ (not square-integrable!) such that $\Delta_0 f = \lambda f$ [31, Theorem 2.1]. However, this contradicts the parabolicity of M . \square

We do not know of any result analogous to Proposition 10 for general two-dimensional simplicial complexes, say uniformly finite. See, however, Subsection 5.2.