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2. If M is an irreducible noncompact globally symmetric space G/K , with $G = \text{Isom}(M)$ and K a maximal compact subgroup, then one can say more about the bottom of the spectrum. If $\text{rk}(G) = \text{rk}(K)$ then $\text{Ker} \left(\Delta_{\frac{\dim(M)}{2}} \right)$ is infinite-dimensional and the spectrum of Δ is bounded away from zero otherwise. If $\text{rk}(G) > \text{rk}(K)$ then $\text{Ker}(\Delta) = 0$ and $0 \in \sigma(\Delta_p)$ if and only if

$$p \in \left[\frac{\dim(M)}{2} - \frac{\text{rk}(G) - \text{rk}(K)}{2}, \frac{\dim(M)}{2} + \frac{\text{rk}(G) - \text{rk}(K)}{2} \right]$$

[19, Section VIIB].

Finally, we state a result about uniformly contractible Riemannian manifolds.

DEFINITION 6 [15, p. 29]. *A metric space Z has finite asymptotic dimension if there is an integer n such that for any $r > 0$, there is a covering $Z = \bigcup_{i \in I} C_i$ of Z by subsets of uniformly bounded diameter so that no metric ball of radius r in Z intersects more than $n + 1$ elements of $\{C_i\}_{i \in I}$. The smallest such integer n is called the asymptotic dimension $\text{asdim}_+(Z)$ of Z .*

PROPOSITION 8 (Yu [33]). *If M is a uniformly contractible Riemannian manifold with finite asymptotic dimension then $0 \in \sigma(\Delta_p)$ for some p .*

The proof of Proposition 8 uses methods of coarse index theory [28].

4. VERY LOW DIMENSIONS

In this section we show that the answer to the zero-in-the-spectrum question is “yes” for one-dimensional simplicial complexes and two-dimensional Riemannian manifolds.

4.1 ONE DIMENSION

As a one-dimensional manifold is either S^1 or \mathbf{R} , zero is clearly in the spectrum.

A more interesting problem is to consider a connected one-dimensional simplicial complex K . Let V be the set of vertices of K and let E be the set of oriented edges of K . That is, an element e of E consists of an edge of K and an ordering (s_e, t_e) of ∂e . We let $-e$ denote the same edge with the

reverse ordering of ∂e . For $x \in V$, let m_x denote the number of unoriented edges of which x is a boundary. We assume that $m_x < \infty$ for all x . Put

$$(4.1) \quad \begin{aligned} C^0(K) &= \{f : V \rightarrow \mathbf{C} \text{ such that } \sum_{x \in V} m_x |f(x)|^2 < \infty\}, \\ C^1(K) &= \{F : E \rightarrow \mathbf{C} \text{ such that} \\ &\quad F(-e) = -F(e) \text{ and } \frac{1}{2} \sum_{e \in E} |F(e)|^2 < \infty\}. \end{aligned}$$

Then $C^0(K)$ and $C^1(K)$ are Hilbert spaces. The weighting used to define $C^0(K)$ is natural in certain respects [8].

There is a bounded operator $d : C^0(K) \rightarrow C^1(K)$ given by $(df)(e) = f(t_e) - f(s_e)$. Define the Laplace-Beltrami operators by $\Delta_0 = d^*d$ and $\Delta_1 = dd^*$. An element of $\text{Ker}(\Delta_1)$ is an $F \in C^1(K)$ such that for each vertex x the total current flowing into x vanishes, i.e. $\sum_{e \in E: t_e = x} F(e) = 0$.

The next proposition is essentially due to Gromov [15, p. 236], who proved it in the case when $\{m_x\}_{x \in V}$ is bounded.

PROPOSITION 9. $0 \in \sigma(\Delta_0)$ or $0 \in \sigma(\Delta_1)$.

Proof. As the nonzero spectra of d^*d and dd^* are the same, for our purposes it suffices to consider $\sigma(\Delta_0)$ and $\text{Ker}(\Delta_1)$. We argue by contradiction. Suppose that $0 \notin \sigma(\Delta_0)$ and $\text{Ker}(\Delta_1) = 0$. First, K must be infinite, as otherwise $\text{Ker}(\Delta_0) \neq 0$. Second, K must be a tree, as if K had a loop then we could create a nonzero element of $\text{Ker}(\Delta_1)$ by letting a current of unit strength flow around the loop.

We now show that K has lots of branching. For $x, y \in V$, let $[x, y]$ be the geodesic arc from x to y and let (x, y) be its interior. Let $d(x, y)$ be the number of edges in $[x, y]$.

LEMMA 5. *There is a constant $L > 0$ such that if $d(x, y) > L$ then there is an infinite subtree of K which intersects (x, y) but does not contain x or y .*

Proof. If the lemma is not true then for all $N > 1$, there are vertices x and y such that $d(x, y) > N$ but there are no infinite subtrees as in the statement of the lemma. In other words, the connected component C of $K - \{x\} - \{y\}$ which contains (x, y) is finite. As K is a tree, x is only connected to the vertices in C by a single edge, and similarly for y (see Fig. 5). Define $f \in C^0(K)$ by

$$(4.2) \quad f(v) = \begin{cases} 1 & \text{if } v \in C, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$(4.3) \quad \frac{\langle df, df \rangle}{\langle f, f \rangle} \leq \frac{2}{2(d(x, y) - 1)} \leq \frac{1}{N}.$$

As N can be taken arbitrarily large, this contradicts the assumption that $0 \notin \sigma(\Delta_0)$. \square

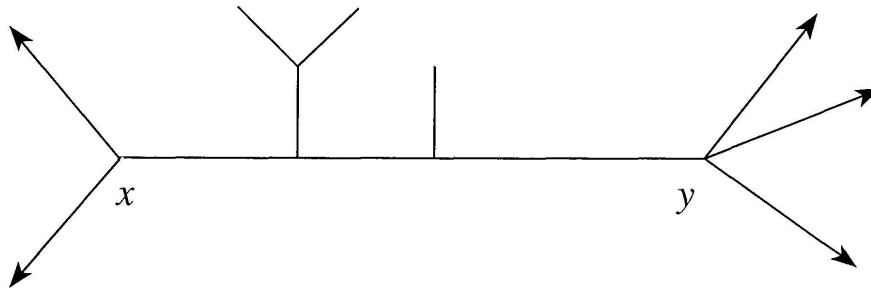


FIGURE 5

It follows that K contains a subtree K' which is topologically equivalent to an infinite triadic tree, with the distances between branchings at most L (see Fig. 6). We can create a nonzero square-integrable harmonic 1-cochain F' on K' by letting a unit current flow through it, as in Fig. 6. Let $F \in C^1(K)$ be the extension of F' by zero to K . If x is a vertex of K' then the total current flowing into x is still zero, as no new current comes in along the edges of $K - K'$. Thus $\text{Ker}(\Delta_1) \neq 0$, which is a contradiction. \square

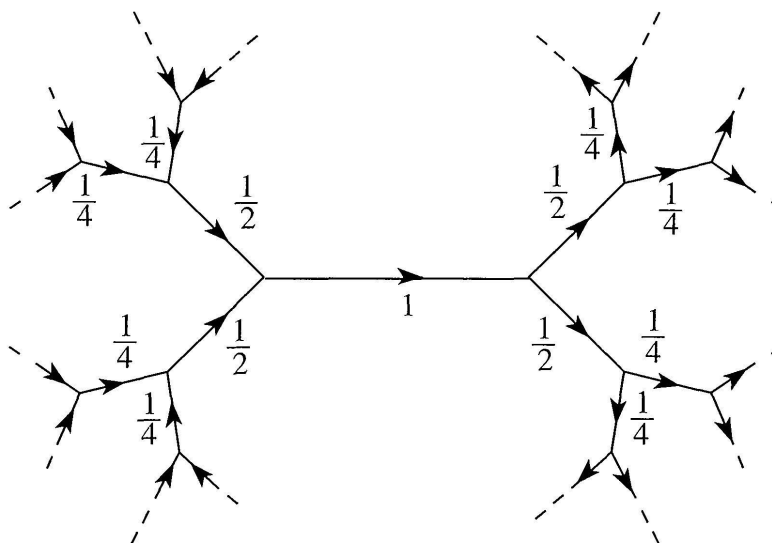


FIGURE 6

4.2 TWO DIMENSIONS

PROPOSITION 10 (Lott, Dodziuk). *The answer to the zero-in-the-spectrum question is “yes” if M is a two-dimensional manifold.*

Proof. The Hodge decomposition gives

$$(4.4) \quad \Lambda^0(M) = \text{Ker}(\Delta_0) \oplus \Lambda^0(M) / \text{Ker}(d),$$

$$(4.5) \quad \Lambda^1(M) = \text{Ker}(\Delta_1) \oplus \overline{d\Lambda^0(M)} \oplus \overline{*d\Lambda^0(M)},$$

$$(4.6) \quad \Lambda^2(M) = *\text{Ker}(\Delta_0) \oplus *(\Lambda^0(M) / \text{Ker}(d)).$$

Thus it is enough to look at

$$\text{Ker}(\Delta_0), \quad \text{Ker}(\Delta_1) \quad \text{and} \quad \sigma(\Delta_0 \text{ on } \Lambda^0(M) / \text{Ker}(d)).$$

We argue by contradiction. Assume that zero is not in the spectrum. By Proposition 4, $\text{Im}(H_c^1(M) \rightarrow H^1(M)) = 0$. Thus M must be planar, in the sense of either of the following two equivalent conditions :

1. Any simple closed curve in M separates it into two pieces.
2. M is diffeomorphic to the complement of a closed subset of S^2 .

As $\text{Ker}(\Delta_0) = 0$, M cannot be S^2 . By Proposition 5, the possible existence of nonzero square-integrable harmonic 1-forms on M only depends on the underlying Riemann surface coming from the Riemannian metric on M .

We recall some notions from Riemann surface theory [1]. A function $f \in C^\infty(M)$ is *superharmonic* if $\Delta_0 f > 0$. (This is a conformally-invariant statement.) The Riemann surface underlying M is *hyperbolic* if it has a positive superharmonic function and *parabolic* otherwise. If M is planar and hyperbolic then there is a nonconstant harmonic function $f \in C^\infty(M)$ such that $\int_M df \wedge *df < \infty$ [1, p. 208]. Then df would be a nonzero element of $\text{Ker}(\Delta_1)$. Thus M must be parabolic.

Put $\lambda_0 = \inf(\sigma(\Delta_0))$. Choose some λ such that $0 < \lambda < \lambda_0$. Then there is a positive $f \in C^\infty(M)$ (not square-integrable !) such that $\Delta_0 f = \lambda f$ [31, Theorem 2.1]. However, this contradicts the parabolicity of M . \square

We do not know of any result analogous to Proposition 10 for general two-dimensional simplicial complexes, say uniformly finite. See, however, Subsection 5.2.