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DEFINITION 3. We say that $H^j(M; \mathbf{C})$ vanishes uniformly if for all $r > 0$, there is an $R(r) \geq r$ such that for all $m \in M$,

$$(2.11) \quad \text{Im}(H^j(B_{R(r)}(m); \mathbf{C}) \rightarrow H^j(B_r(m); \mathbf{C})) = 0.$$

PROPOSITION 3 (Pansu [25]). Consider a Riemannian manifold M of bounded geometry such that for some $k > 0$, $H^j(M; \mathbf{C})$ vanishes uniformly for $1 \leq j \leq k$. Then within the class of such manifolds,

1. $\bar{H}_{(2)}^p(M)$ and $H_{(2)}^p(M)$ are coarse quasi-isometry invariants for $0 \leq p \leq k$.
2. $\text{Ker}(\bar{H}_{(2)}^{k+1}(M) \rightarrow H^{k+1}(M; \mathbf{C}))$ and $\text{Ker}(H_{(2)}^{k+1}(M) \rightarrow H^{k+1}(M; \mathbf{C}))$ are coarse quasi-isometry invariants.

3. GENERAL PROPERTIES OF L^2 -COHOMOLOGY

In this section we give some general results about the L^2 -cohomology of complete Riemannian manifolds. First, we give a useful sufficient condition for the reduced L^2 -cohomology to be nonzero.

PROPOSITION 4. For all p , $\text{Im}(H_c^p(M; \mathbf{C}) \rightarrow H^p(M; \mathbf{C}))$ injects into $\bar{H}_{(2)}^p(M)$.

Proof. Suppose that ω is a smooth compactly-supported closed p -form which represents a nonzero class in $H^p(M; \mathbf{C})$. By Poincaré duality, there is a smooth compactly-supported closed $(\dim(M) - p)$ -form ρ such that $\int_M \omega \wedge \rho \neq 0$.

As ω is compactly-supported, it is square-integrable and so represents an element $[\omega]$ of $\bar{H}_{(2)}^p(M)$. Suppose that $[\omega] = 0$. Then there is a sequence $\{\eta_i\}_{i=1}^\infty$ in $\Omega^{p-1}(M)$ such that $\omega = \lim_{i \rightarrow \infty} d\eta_i$, where the limit is in an L^2 -sense. It follows that

$$(3.1) \quad \int_M \omega \wedge \rho = \lim_{i \rightarrow \infty} \int_M d\eta_i \wedge \rho = \lim_{i \rightarrow \infty} \int_M d(\eta_i \wedge \rho) = 0,$$

which is a contradiction. Thus $[\omega] \neq 0$. \square

COROLLARY 2. *Let N^{4k} be a compact manifold-with-boundary with nonzero signature. Then if M is any complete Riemannian manifold which is diffeomorphic to $\text{int}(N)$, $\overline{H}_{(2)}^{2k}(M) \neq 0$.*

Proof. By definition, the signature of N is the signature of the intersection form on

$$(3.2) \quad \text{Im} (H^{2k}(N, \partial N; \mathbf{C}) \rightarrow H^{2k}(N; \mathbf{C})) \cong \text{Im} (H_c^{2k}(M; \mathbf{C}) \rightarrow H^{2k}(M; \mathbf{C})) .$$

If the signature of N is nonzero then $\text{Im} (H_c^{2k}(M; \mathbf{C}) \rightarrow H^{2k}(M; \mathbf{C}))$ must be nonzero. The corollary follows from Proposition 4. \square

EXAMPLE. Let N be $\mathbf{C}P^2$ with a small 4-ball removed. Then N satisfies the hypothesis of Corollary 2.

We now show that the middle-dimensional reduced L^2 -cohomology is a conformal invariant of M .

PROPOSITION 5. *If M^{2k} is even-dimensional then $\text{Ker}(\Delta_k)$ is conformally-invariant.*

Proof. Suppose that g and $e^\phi g$ are conformally equivalent Riemannian metrics on M , with $\phi \in C^\infty(M)$. We use the fact that the action of the Hodge duality operator $*$ on $\Lambda^k(M)$ is independent of ϕ . If ω is a k -form on M , its L^2 -norm $\int_M \omega \wedge *\omega$ is independent of ϕ . Thus the Hilbert space $\Lambda^k(M)$ is independent of ϕ . Furthermore,

$$(3.3) \quad \begin{aligned} \text{Ker}(\Delta_k) &= \{\omega \in \Lambda^k(M) : d\omega = d^*\omega = 0\} \\ &= \{\omega \in \Lambda^k(M) : d\omega = \pm * d * (\omega) = 0\} \end{aligned}$$

$$(3.4) \quad = \{\omega \in \Lambda^k(M) : d\omega = d * (\omega) = 0\}$$

is independent of ϕ . \square

EXAMPLE. Take $M = H^2$. Then M is conformally equivalent to a Euclidean disk D . The harmonic square-integrable 1-forms on D are of the form $f_1(x, y) dx + f_2(x, y) dy$, where f_1 and f_2 are square-integrable harmonic functions on D . There is clearly an infinite number of such functions, and so $\dim(\overline{H}_{(2)}^1(H^2)) = \infty$. The same argument applies to H^{2k} , to give $\dim(\overline{H}_{(2)}^k(H^{2k})) = \infty$.

In the case of functions, one has a good control of when zero is in the spectrum of the Laplacian.

LEMMA 4. $\text{Ker}(\Delta_0) \neq 0$ if and only if $\text{vol}(M) < \infty$.

Proof. If $\text{vol}(M) < \infty$ then the constant functions on M are square-integrable and harmonic. Conversely, if $f \in \text{Ker}(\Delta_0)$ then by Lemma 2, f is constant. If f is nonzero and square-integrable then $\text{vol}(M) < \infty$.

DEFINITION 4. M is open at infinity if there is a constant $C > 0$ such that for all domains D in M with smooth compact closure, $\frac{\text{area}(\partial D)}{\text{vol}(D)} \geq C$.

EXAMPLES.

1. \mathbf{R}^n is not open at infinity, as can be seen by taking large balls for D .
2. H^n is open at infinity.

PROPOSITION 6 (Buser [3]). Let M have infinite volume. Suppose that there is a constant $c \geq 0$ such that $\text{Ricci}_M \geq -c^2$. Then $0 \notin \sigma(\Delta_0)$ if and only if M is open at infinity.

Proof.

1. Suppose that M is open at infinity. By Cheeger's inequality,

$$(3.5) \quad \inf(\sigma(\Delta_0)) \geq \inf_D \frac{1}{4} \left(\frac{\text{area}(\partial D)}{\text{vol}(D)} \right)^2 > 0.$$

2. Suppose that M is not open at infinity. The bottom of the spectrum of Δ_0 is given in terms of Rayleigh quotients by

$$(3.6) \quad \inf(\sigma(\Delta_0)) = \inf_{f \neq 0} \frac{\int_M |df|^2}{\int_M f^2},$$

where f ranges over compactly-supported Lipschitz functions on M . We want to find compactly-supported Lipschitz functions on M of arbitrarily small Rayleigh quotient. By assumption, for all $\epsilon > 0$ there is a domain D such that $\frac{\text{area}(\partial D)}{\text{vol}(D)} \leq \epsilon$. Put

$$(3.7) \quad N_1(\partial D) = \{m \in M : m \notin D \text{ and } d(m, \partial D) \leq 1\}.$$

Define a function f , which approximates the characteristic function of D , by

$$(3.8) \quad f(m) = \begin{cases} 1 & \text{if } m \in D \\ 1 - d(m, \partial D) & \text{if } m \in N_1(\partial D) \\ 0 & \text{if } m \notin D \text{ and } m \notin N_1(\partial D). \end{cases}$$

Clearly $\int_M f^2 \geq \text{vol}(D)$. As f has nonzero gradient only in $N_1(\partial D)$, where $|df| = 1$ almost everywhere, we have $\int_M |df|^2 = \text{vol}(N_1(\partial D))$. If D is nice and round then we expect that

$$(3.9) \quad \text{vol}(N_1(\partial D)) \sim \text{area}(\partial D)$$

and the Rayleigh quotient $\frac{\int_M |df|^2}{\int_M f^2}$ will be comparable to ϵ .

The only problem with this argument is that D may not be nice and round, but may have long thin legs coming out of it, like an octopus. Then (3.9) may not be valid. The content of [3] is that if this is the case, we can cut the legs off of D to come up with a new domain for which the above heuristic argument is valid. We refer to [3] for details. \square

COROLLARY 3 (Brooks [2]). *Let M be a normal covering of a compact manifold X with covering group Γ . Then $0 \in \sigma(\Delta_0)$ on M if and only if Γ is amenable.*

Proof. If Γ is finite then $0 \in \sigma(\Delta_0)$ and Γ is amenable. If Γ is infinite then by Proposition 6, $0 \in \sigma(\Delta_0)$ if and only if M is not open at infinity. Let S be a finite set of generators of Γ . Let G be the Cayley graph of Γ , constructed using S . There is a notion of G being open at infinity which is similar to Definition 4. As M is coarsely quasi-isometric to G , M is not open at infinity if and only if G is not open at infinity. However, this is one of the characterizations of amenability of Γ . \square

We now prove a result about manifolds which, roughly speaking, are at least as large as Euclidean space.

DEFINITION 5. *M is hyperEuclidean if there is a proper distance-nonincreasing map $F : M \rightarrow \mathbf{R}^{\dim(M)}$ of nonzero degree.*

REMARKS.

1. A map is proper if preimages of compact sets are compact. Instead of requiring that F be distance-nonincreasing, we could require that F have a finite Lipschitz constant. By postcomposing F with a dilatation of $\mathbf{R}^{\dim(M)}$, the two conditions are equivalent.
2. If M is hyperEuclidean then a compactly-supported modification of M is also hyperEuclidean.

3. Examples of hyperEuclidean manifolds are given by simply-connected nonpositively-curved manifolds M . Namely, fix $m_0 \in M$ and put $F = \exp_{m_0}^{-1}$.
4. There was once a conjecture that all uniformly contractible manifolds are hyperEuclidean (with a degree-one map to $\mathbf{R}^{\dim(M)}$), but this turns out to be wrong [11]. There is still an open conjecture that a uniformly contractible manifold of bounded geometry is hyperEuclidean, and in particular, that the universal cover of an aspherical closed manifold is hyperEuclidean.

PROPOSITION 7 (Gromov [15, p. 238]). *If M is hyperEuclidean then $0 \in \sigma(\Delta_p)$ for some p .*

Proof. Put $n = \dim(M)$. First, suppose that n is even. We will construct a vector bundle E with connection on \mathbf{R}^n which is topologically nontrivial but analytically trivial, in a sense which will be made precise. Then assuming that zero is not in the spectrum of M , we will apply the relative index theorem to F^*E in order to get a contradiction.

Recall that $K^0(S^n) = \mathbf{Z} \oplus \mathbf{Z}$. If \mathcal{E} is a (virtual) vector bundle on S^n , the two \mathbf{Z} factors correspond to $\text{rk}(\mathcal{E})$ and $\int_{S^n} \text{ch}(\mathcal{E})$, respectively. This means that for some $N > 0$, there is a \mathbf{C}^N -bundle \mathcal{E} on S^n with $\int_{S^n} \text{ch}(\mathcal{E}) \neq 0$. Fixing a point $\infty \in S^n$, we can trivialize \mathcal{E} in a neighborhood of ∞ . Furthermore, we can put a Hermitian metric and Hermitian connection on \mathcal{E} so that the connection is flat in a neighborhood of ∞ .

Let E be the restriction of \mathcal{E} to $\mathbf{R}^n = S^n - \{\infty\}$. Let ∇ be the restriction of the Hermitian connection on \mathcal{E} to \mathbf{R}^n . Then E is trivialized outside of a compact set $K \subset \mathbf{R}^n$ and ∇ is flat outside of K .

As \mathbf{R}^n is contractible, there is an isomorphism of Hermitian vector bundles $i : \mathbf{R}^n \times \mathbf{C}^N \rightarrow E$. Then $i^*\nabla$ can be considered to be a $u(N)$ -valued 1-form ω on \mathbf{R}^n . The curvature of ω is the $u(N)$ -valued 2-form $\Omega = d\omega + \omega^2$. The nontriviality of \mathcal{E} translates to the facts that

1. Ω vanishes outside of K and
2. The de Rham cohomology class of the compactly-supported form

$$\text{Tr} \left(e^{-\frac{\Omega}{2\pi i}} \right) - N$$

is a nonzero multiple of the fundamental class $[\mathbf{R}^n] \in H_c^n(\mathbf{R}^n; \mathbf{R})$.

In fact, we can take ω to have a finite L^∞ -norm $\|\omega\|_\infty$. For example, if $n = 2$, take $N = 1$. Let $f \in C_0^\infty([0, \infty))$ be a nonincreasing function such that if $x \in [0, 1]$ then $f(x) = 1$. Put $\omega = -i(1 - f(r)) d\theta$. Then

$$(3.10) \quad \Omega = d\omega = if'(r)dr \wedge d\theta.$$

We have $\|\omega\|_\infty = \sup_{r>0} \frac{1-f(r)}{r}$ and $\int_{\mathbf{R}^2} \left[\text{Tr} \left(e^{-\frac{\Omega}{2\pi i}} \right) - 1 \right] = 1$.

Returning to the case of general even n , for $\epsilon > 0$, let $\Phi_\epsilon : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be the map $\Phi_\epsilon(\mathbf{x}) = \epsilon\mathbf{x}$. Put $\omega_\epsilon = \Phi_\epsilon^*\omega$ and $\Omega_\epsilon = d\omega_\epsilon + \omega_\epsilon^2$. Then

$$(3.11) \quad \begin{aligned} \|\omega_\epsilon\|_\infty &= \epsilon \|\omega\|_\infty \quad \text{and} \quad \int_{\mathbf{R}^n} \left[\text{Tr} \left(e^{-\frac{\Omega_\epsilon}{2\pi i}} \right) - N \right] \\ &= \int_{\mathbf{R}^n} \left[\text{Tr} \left(e^{-\frac{\Omega}{2\pi i}} \right) - N \right] \neq 0. \end{aligned}$$

We now turn our attention to M . Suppose that $0 \notin \sigma(\Delta_p)$ for all p . Consider the self-adjoint operator $d + d^*$ on $\Lambda^*(M)$. As $(d + d^*)^2 = \Delta$, it follows that $0 \notin \sigma(d + d^*)$. In other words, $d + d^*$ is L^2 -invertible. Define an operator μ on $\Lambda^*(M)$ by saying that if $\omega \in \Lambda^p(M)$ then

$$(3.12) \quad \mu(\omega) = i^{\frac{n(n-1)}{2}} (-1)^{\frac{p(p-1)}{2}} * (\omega).$$

One can check that $\mu^2 = 1$ and $\mu(d + d^*) + (d + d^*)\mu = 0$.

Clearly the operator $(d + d^*) \otimes \text{Id}_N$, acting on $\Lambda^*(M) \otimes \mathbf{C}^N$, is also invertible. Consider the $u(N)$ -valued 1-form $F^*\omega_\epsilon$ on M . As F is distance-nonincreasing,

$$(3.13) \quad \|F^*\omega_\epsilon\|_\infty \leq \|\omega_\epsilon\|_\infty = \epsilon \|\omega\|_\infty.$$

Let $e(F^*\omega_\epsilon)$ denote exterior multiplication by $F^*\omega_\epsilon$, acting on $\Lambda^*(M) \otimes \mathbf{C}^N$ and let $i(F^*\omega_\epsilon)$ denote interior multiplication by $F^*\omega_\epsilon$. By making ϵ small enough, the operator $e(F^*\omega_\epsilon) - i(F^*\omega_\epsilon)$ has arbitrarily small norm and so the operator $((d + d^*) \otimes \text{Id}_N) + e(F^*\omega_\epsilon) - i(F^*\omega_\epsilon)$ is also invertible.

Put $D = (d \otimes \text{Id}_N) + e(F^*\omega_\epsilon)$. Then D is exterior differentiation, using the connection $F^*\omega_\epsilon$, and

$$(3.14) \quad D + D^* = ((d + d^*) \otimes \text{Id}_N) + e(F^*\omega_\epsilon) - i(F^*\omega_\epsilon).$$

As $(d + d^*) \otimes \text{Id}_N$ and $D + D^*$ anticommute with $\mu \otimes \text{Id}_N$, they have well-defined indices which happen to vanish, as the operators are invertible. On the other hand, let $L(M)$ be the Hirzebruch L -form. The relative index theorem of Gromov and Lawson [10, 16] says that

$$(3.15) \quad \begin{aligned} \text{ind}(D + D^*) - \text{ind}((d + d^*) \otimes \text{Id}_N) \\ = \int_M L(M) \wedge \left[\text{Tr} \left(e^{-\frac{F^*\Omega_\epsilon}{2\pi i}} \right) - N \right]. \end{aligned}$$

As F is proper, the de Rham cohomology class of $\text{Tr} \left(e^{-\frac{F^* \Omega_\epsilon}{2\pi i}} \right) - N = F^* \left[\text{Tr} \left(e^{-\frac{\Omega_\epsilon}{2\pi i}} \right) - N \right]$ is well-defined as a multiple of the fundamental class $[M] \in H_c^n(M; \mathbf{R})$. As the series for $L(M)$ starts off as $L(M) = 1 + \dots$, we obtain

$$\begin{aligned}
 \text{ind}(D + D^*) - \text{ind}((d + d^*) \otimes \text{Id}_N) &= \int_M \left[\text{Tr} \left(e^{-\frac{F^* \Omega_\epsilon}{2\pi i}} \right) - N \right] \\
 (3.16) \qquad \qquad \qquad &= \int_M F^* \left[\text{Tr} \left(e^{-\frac{\Omega_\epsilon}{2\pi i}} \right) - N \right] \\
 &= \text{deg}(F) \int_{\mathbf{R}^n} \left[\text{Tr} \left(e^{-\frac{\Omega_\epsilon}{2\pi i}} \right) - N \right] \neq 0.
 \end{aligned}$$

This contradicts the vanishing of $\text{ind}(D + D^*)$ and $\text{ind}((d + d^*) \otimes \text{Id}_N)$. Thus zero must be in the spectrum of M after all.

Now suppose that n is odd. As M is hyperEuclidean, so is $\mathbf{R} \times M$. With respect to the decomposition $\Lambda^*(\mathbf{R} \times M) = \Lambda^*(\mathbf{R}) \otimes \Lambda^*(M)$, the Laplace-Beltrami operator on $\mathbf{R} \times M$ decomposes as

$$(3.17) \qquad \Delta_{\mathbf{R} \times M} = (\Delta_{\mathbf{R}} \otimes I) + (I \otimes \Delta_M).$$

Then

$$(3.18) \qquad \sigma(\Delta_{\mathbf{R} \times M}) = \{ \lambda_1 + \lambda_2 : \lambda_1 \in [0, \infty) \text{ and } \lambda_2 \in \sigma(\Delta_M) \}.$$

From what has already been proved, $0 \in \sigma(\Delta_{\mathbf{R} \times M})$. It follows that $0 \in \sigma(\Delta_M)$. \square

REMARKS.

1. We have shown that if M is hyperEuclidean then $0 \in \sigma(\Delta_p)$ for some p . One can ask whether the number p can be pinned down. In general, when computing the index of the operator $d + d^*$, the differential forms outside of the middle dimensions do not contribute. This is a reflection of the fact that the signature of a closed manifold can be computed using only the middle-dimensional cohomology. So this gives some reason to think that if $\dim(M)$ is even then $0 \in \sigma \left(\Delta_{\frac{\dim(M)}{2}} \right)$.

Unfortunately, the operator $(D + D^*)^2$ does not preserve the degree of a differential form and so we cannot use the above proof to reach the desired conclusion. However, with a more refined index theorem [28, Theorem 6.24], one can indeed conclude that $0 \in \sigma \left(\Delta_{\frac{\dim(M)}{2}} \right)$ if $\dim(M)$ is even and that $0 \in \sigma \left(\Delta_{\frac{\dim(M) \pm 1}{2}} \right)$ if $\dim(M)$ is odd.

2. If M is an irreducible noncompact globally symmetric space G/K , with $G = \text{Isom}(M)$ and K a maximal compact subgroup, then one can say more about the bottom of the spectrum. If $\text{rk}(G) = \text{rk}(K)$ then $\text{Ker} \left(\Delta_{\frac{\dim(M)}{2}} \right)$ is infinite-dimensional and the spectrum of Δ is bounded away from zero otherwise. If $\text{rk}(G) > \text{rk}(K)$ then $\text{Ker}(\Delta) = 0$ and $0 \in \sigma(\Delta_p)$ if and only if

$$p \in \left[\frac{\dim(M)}{2} - \frac{\text{rk}(G) - \text{rk}(K)}{2}, \frac{\dim(M)}{2} + \frac{\text{rk}(G) - \text{rk}(K)}{2} \right]$$

[19, Section VIIB].

Finally, we state a result about uniformly contractible Riemannian manifolds.

DEFINITION 6 [15, p. 29]. *A metric space Z has finite asymptotic dimension if there is an integer n such that for any $r > 0$, there is a covering $Z = \bigcup_{i \in I} C_i$ of Z by subsets of uniformly bounded diameter so that no metric ball of radius r in Z intersects more than $n + 1$ elements of $\{C_i\}_{i \in I}$. The smallest such integer n is called the asymptotic dimension $\text{asdim}_+(Z)$ of Z .*

PROPOSITION 8 (Yu [33]). *If M is a uniformly contractible Riemannian manifold with finite asymptotic dimension then $0 \in \sigma(\Delta_p)$ for some p .*

The proof of Proposition 8 uses methods of coarse index theory [28].

4. VERY LOW DIMENSIONS

In this section we show that the answer to the zero-in-the-spectrum question is “yes” for one-dimensional simplicial complexes and two-dimensional Riemannian manifolds.

4.1 ONE DIMENSION

As a one-dimensional manifold is either S^1 or \mathbf{R} , zero is clearly in the spectrum.

A more interesting problem is to consider a connected one-dimensional simplicial complex K . Let V be the set of vertices of K and let E be the set of oriented edges of K . That is, an element e of E consists of an edge of K and an ordering (s_e, t_e) of ∂e . We let $-e$ denote the same edge with the