

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 42 (1996)  
**Heft:** 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** THE ZERO-IN-THE-SPECTRUM QUESTION  
**Autor:** LOTT, John  
**Kapitel:** 1. Introduction  
**DOI:** <https://doi.org/10.5169/seals-87882>

#### Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

#### Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

#### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 19.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## THE ZERO-IN-THE-SPECTRUM QUESTION

by John LOTT

ABSTRACT. This is an expository article on the question of whether zero lies in the spectrum of the Laplace-Beltrami operator acting on differential forms on a manifold.

### 1. INTRODUCTION

Let  $M$  be a complete connected oriented Riemannian manifold. The Laplace-Beltrami operator  $\Delta_p$  acts on the square-integrable  $p$ -forms on  $M$ . We asked the following question in 1991 :

**ZERO-IN-THE-SPECTRUM QUESTION.** Is zero always in the spectrum of  $\Delta_p$  for some  $p$  ?

To our knowledge, nobody has found a counterexample. The question was also raised by Gromov in the case of a contractible manifold with a discrete cocompact group of isometries ([15], p. 21).

Being able to answer the above question is a first step toward understanding the spectrum of the Laplace-Beltrami operator. We would also like to be able to say whether or not zero is in the spectrum of  $\Delta_p$  for a given  $p$ . This problem is partly topological in nature and partly geometric, in a sense which will be made precise later. In fact, it is equivalent to knowing the (unreduced)  $L^2$ -cohomology of  $M$ . The study of  $L^2$ -cohomology touches on many branches of mathematics, including combinatorial group theory, topology, differential geometry and algebraic geometry. It is most commonly considered when  $M$  is the universal cover of a compact manifold or when  $M$  is a finite-volume

Hermitian locally symmetric space. We refer to [22, 26] and [30] for surveys of these two cases. In this article we will instead emphasize general complete Riemannian manifolds and give some partial positive answers to the zero-in-the-spectrum question.

The sections of the article are :

1. Introduction
2. Definition of  $L^2$ -Cohomology
3. General Properties of  $L^2$ -Cohomology
4. Very Low Dimensions
  - 4.1. One Dimension
  - 4.2. Two Dimensions
5. Universal Covers
  - 5.1. Big and Small Groups
  - 5.2. Two and Three Dimensions
  - 5.3. Four Dimensions
  - 5.4. More Dimensions
6. Topologically Tame Manifolds

In what follows, all manifolds will be smooth, connected, oriented and of positive dimension. All maps between manifolds will be orientation-preserving. Unless otherwise indicated, all Riemannian manifolds will be complete.

We have tried to give as many complete proofs as reasonably possible. All unattributed results are of unknown origin or are due to the author. I thank Wolfgang Lück for conversations on some of the topics discussed herein. I thank Marie-Claude Vergne for making the figures. This article is based on lectures given at the Troisième Cycle Romand “On the Conjecture of the Zero in the Spectrum” held at Les Diablerets, Switzerland, March, 1996. I warmly thank Alain Valette and the other organizers and participants of the meeting.