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α_5 and cap off the resulting pair of boundary curves with two disks, then we have a realization of the corresponding collection

$$S' = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \alpha_3 + \alpha_4\}$$

contracted down onto a surface of genus 4. This collection is definitely realizable. In particular we have a corresponding family of 4 pairwise disjoint simple closed curves on the 2×4-punctured sphere. Homology considerations show that the pair of disks, which must be removed, with the resulting boundaries identified, to re-construct the original surface of genus 5, both lie on the same side of the curves C_2 realizing $\gamma_2 = \alpha_1 + \alpha_2 + \alpha_3$ and C_3 realizing $\gamma_3 = \alpha_2 + \alpha_3 + \alpha_4$. But by bare hands one can show that for any realization of $\gamma_1 = \alpha_1 + \alpha_2$, γ_2, γ_3, and $\gamma_4 = \alpha_3 + \alpha_4$ by pairwise disjoint simple closed curves on the 2×4-punctured sphere, both of the curves C_1 and C_4 giving the classes γ_1 and γ_4 must be separated by both of the curves C_2 and C_3. But because our particular realization comes from a hypothesized realization of curves on a genus 5 surface, the added disks above must lie on the same side of C_2 as does C_1 *and also* as does C_4. This contradiction shows that the given collection cannot be realized.

REMARK. *The same set of 9 homology classes gives an example in* any *surface of genus greater than 5 of homology classes satisfying the Intersection, Summand, and Size Conditions that cannot be realized by a corresponding family of pairwise disjoint simple closed curves.*

This follows from Theorem 7.4 and Theorem 8.1 below.

8. SOME FINAL OBSERVATIONS

Notice that the Intersection, Summand, and Size Conditions in Theorem 1 make no mention of the genus of the ambient surface. A natural thought is that these three conditions might suffice to realize given homology classes by pairwise disjoint simple closed curves provided one is allowed to "stabilize" the surface by adding additional handles. Here we show that there is nothing gained by such stabilization.

PROPOSITION 8.1. *Suppose a surface F is expressed as a connected sum $F_1 \# F_2$ and we identify $H_1(F) = H_1(F_1) \oplus H_1(F_2)$ in the obvious way. Suppose further $S \subset H_1(F_1) \subset H_1(F)$ is a family of homology classes that can be realized by pairwise disjoint simple closed curves in F. Then S can be realized by pairwise disjoint simple closed curves in F_1.*

Proof. Suppose F_i has genus g_i, so that F has genus $g = g_1 + g_2$. As usual we let $S = \{\alpha_1, \dots, \alpha_n, \gamma_1, \dots, \gamma_k\}$, where $\alpha_1, \dots, \alpha_n$ form a basis for span S. In particular, $n \leq g_1$ and $\alpha_1, \dots, \alpha_n$ form part of a symplectic basis for the homology of F_1 as well as for F. We let $A_1, \dots, A_n, C_1, \dots, C_k$ denote pairwise disjoint simple closed curves representing the corresponding homology classes. We let $\widehat{F}$ denote the result of cutting F open along $A_1, \dots, A_n$ and $\overline{F}$ the result of filling in $\widehat{F}$ with $2n$ disks. Then $\overline{F}$ is a closed surface of genus $g - n$. We now view the curves $C_1, \dots, C_k$ as living in $\overline{F}$, but missing the added disks. Note that these curves are all null-homologous in $\overline{F}$ and hence each one of them separates $\overline{F}$ and $\widehat{F}$ into two pieces. The homology classes that the latter curves represent in the original surface F and in F_1 are determined up to sign by the collection of disks in $\overline{F}$ they surround. It follows that the curves $C_1, \dots, C_k$ all together separate $\overline{F}$ (or $\widehat{F}$) into $k+1$ pieces, with total genus $g - n$. In particular we see that there are $g - n$ homology classes $\alpha_{n+1}, \dots, \alpha_g$ represented by pairwise disjoint simple closed curves $A_{n+1}, \dots, A_g$ in $\overline{F}$ disjoint from the original $A_1, \dots, A_n$ and $C_1, \dots, C_k$ such that $\alpha_1, \dots, \alpha_g$ is half of a symplectic basis for the homology of F itself. It follows that if we surger away $A_{g_1+1}, \dots, A_g$, then $\alpha_1, \dots, \alpha_{g_1}$ represents half of a symplectic basis for the homology of the resulting surface F' of genus g_1, and if we identify the curves $A_1, \dots, A_{g_1}$ and $C_1, \dots, C_k$ with their images in F', we see that we have indeed embedded pairwise disjoint simple closed curves in $F' \cong F_1$ representing the corresponding homology classes. The point is that the homology classes γ_i of the C_i are completely determined as linear combinations of the α_j. And up to homeomorphism the curves $A_1, \dots, A_{g_1}$ are determined by representing a basis for a summand of the homology on which the intersection pairing vanishes.

The perspective developed above can also be applied to show that any system of pairwise disjoint homologically distinct simple closed curves can be expanded to a maximal set of $3g - 3$ such curves.

PROPOSITION 8.2. *Suppose that F is a closed, orientable surface of genus g and that S is a family of pairwise distinct homology classes represented by pairwise disjoint simple closed curves. Then S can be extended to a family of $3g - 3$ pairwise distinct homology classes represented by a set of pairwise disjoint simple closed curves, including the given collection of simple closed curves.*

Proof. As usual we let $S = \{\alpha_1, \dots, \alpha_n, \gamma_1, \dots, \gamma_k\}$ be a given set of homology classes represented by a corresponding set of pairwise disjoint simple closed curves, $A_1, \dots, A_n$, $C_1, \dots, C_k$, where $\alpha_1, \dots, \alpha_n$ form a basis for span S. We first argue that we can assume that $n = g$. If not, then as above some component of F cut open along all the given curves has positive genus. In that component we can then find a simple closed curve representing a homology class independent of those in S. In this way we increase the span of S until its rank is the maximum possible, namely g.

Now, when we cut open along our expanded family of simple closed curves all resulting components have genus 0. If all components have exactly three boundary components, then the euler characteristic argument of Section 2 shows that our collection already contains $3g - 3$ elements. Otherwise, some component G is a planar surface with at least $m \geq 4$ boundary components. Now when F is reconstructed starting from G one may think of attaching components of $F - G$ to G. None of these extra components can have just one boundary curve, since such a curve would be null-homologous. If such an extra component has two boundary curves, then the corresponding boundary curves of G would not be distinct, so we should actually be thinking in this case of simply identifying the two boundary curves. Suppose that some pair of boundary curves of G is identified in this way. Then it follows that in F the corresponding curve has a dual curve missing all the other curves representing elements of S. In particular that boundary curve of G represents a homology class in F independent of all the other classes in S. Now choose a simple closed curve in G that surrounds one of these two boundary curves and one other boundary curve. It follows that the corresponding homology class is distinct from all other elements of S. In this way we have again expanded the size of S.

Finally we may suppose no pair of boundary curves of G are to be identified. We want to claim that some simple closed curve in G surrounding 3 boundary curves is homologically nontrivial in F and homologically distinct from all other curves so far represented. A typical example of what we are up against is the following: Think of the surface of genus g expressed as the double of a $(g+1)$-holed sphere, with one side further decomposed by more pairwise disjoint, homologically distinct, simple closed curves. Now the challenge is to find more simple closed curves in the second side distinct from those already appearing in the first side. On the first side we have used at most $[3g - 3 - (g+1)]/2 = g - 1$ curves. But on the second side there are, for example, $(g+1)g/2$ different homology classes represented by simple closed curves surrounding just two boundary components.

So suppose G has $m > 3$ boundary curves. Then $H_1(G)$ is free abelian of rank $m-1$, generated by the classes of the boundary curves, with the single relation that the sum of the classes of the boundary curves is 0. Consideration of the other components of $F-G$ then implies additional relations of the form "sum of boundary curves = 0" over the elements in each piece of a partition of the set of boundary components, where each partition piece has at least 3 elements. In particular we can obtain a basis for the image of $H_1(G)$ in $H_1(F)$ by selecting all but one boundary curve from each piece of the partition.

Now in such a surface as G with its m boundary components there are at most $m-3$ pairwise disjoint simple closed curves, pairwise homologically distinct and homologically distinct from the boundary curves. Even if G were not a planar surface, there would be at most $m-3$ such curves homologous to some linear combination of the boundary curves. If the components of $F-G$ are $G_1, \ldots, G_r$, where G_i has m_i boundary curves, then $\sum_{i=1}^{r} m_i = m$. Moreover, the image of $H_1(G)$ in $H_1(F)$ has a basis of $\sum(m_i - 1) = m - r$ elements. Note also that $1 \leq r \leq m/3$, since no component G_i should have just one or two boundary curves. Now in G_i there are at most $m_i - 3$ pairwise disjoint simple closed curves representing homology classes in the linear span of the classes represented by the boundary curves of G_i. It follows that there are already in the originally given collection of curves at most $\sum(m_i - 3) + m = 2m - 3r$ distinct homology classes. On the other hand, within G itself there are some $2^{m-r} - m - 1$ homology classes represented by simple closed curves, excluding the classes represented by the boundary curves and the 0 class. Therefore, in order to expand our originally given collection of simple closed curves by adding a curve inside G, we need to have

$$2^{m-r} - m - 1 > 2m - 3r$$

or

$$2^x - 3x - 1 > 0$$

where $x = m - r$. But $2^x - 3x - 1 \leq 0$ only for $x = 1$ or $x = 2$ (among integral x). That is to say there is trouble only if $m - r = 1$ or $m - r = 2$, i.e., $r = m - 1$ or $m - 2$. But we already noted that we have $1 \leq r \leq m/3$. So, $m - 2 \leq r \leq m/3$, which implies that $m \leq 3$. But we had already seen that we could assume $m > 3$. Thus there must be suitable simple closed curves in G that can be added to the given collection while maintaining the required homological distinctness.