5. Disjoint simple closed curves on a planar surface

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **42 (1996)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **28.04.2024**

of simple closed curves meeting A_n in exactly one point. Now we can band together the components of B', using bands in the complement of A_n to create a closed curve B'' representing γ_n and meeting A_n in exactly one point. But B'' may now have self-intersections. We may then eliminate the self-intersections by sliding segments of B'' over A_n. This creates a simple closed curve B_n meeting A_n in exactly one point, and representing a homology class of the form $\beta_n = \gamma_n + k\alpha_n$, which proves the claim.

Now the union of the two curves A_n and B_n has a small neighborhood N of the form of a once punctured torus. Let F_n denote the result of removing N and replacing it with a disk D. Then $F_n - D = F - N \subset F$ and inclusion identifies $H_1(F_n)$ with the orthogonal complement of α_n and β_n in $H_1(F)$. Thus the homology classes $\alpha_1, \ldots, \alpha_{n-1}$ determine well-defined classes in $H_1(F_n)$, which we continue to call by the same names. By induction there are pairwise disjoint simple closed curves $A_1, \ldots, A_{n-1}$ in F_n representing the homology classes $\alpha_1, \ldots, \alpha_{n-1}$. Then these curves also live in F, determining the same homology classes, and are disjoint from the curve A_n. This completes the proof.

Here is a sketch of a standard but somewhat more learned proof of Proposition 4.2, suggested by M. Kervaire: Any basis for a self-annihilating summand of a skew-symmetric inner product space over **Z** can be extended to be part of a symplectic basis. Any two symplectic bases are related by an isometry of the inner product space. Half of a fixed standard symplectic basis is clearly represented by standard pairwise disjoint simple closed curves in a standard model of the surface. And any isometry is induced by a homeomorphism of the surface, so that the standard curves are taken to the desired curves. To see that any isometry is induced by a homeomorphism one can argue that the symplectic group is generated by certain elementary automorphisms and that these elementary automorphisms are induced by Dehn twist homeomorphisms around standard curves on the surface.

5. Disjoint simple closed curves on a planar surface

Subsequent proofs of realizability of non-independent homology classes will proceed by cutting the surface along curves representing a basis for homology until it becomes a punctured 2-sphere and then representing the remaining homology classes by disjoint curves on this planar surface. We therefore first investigate directly the case of homology classes in a planar surface.

In this section G will denote a compact orientable planar surface with with m oriented boundary components $B_1, \dots, B_m$. By the classification of surfaces G can be thought of as being obtained from the 2-sphere by removing the interiors of m disjoint disks. Now $H_1(G)$ is freely generated by the homology classes $[B_i]$, subject to the single relation $\sum_i [B_i] = 0$.

By the Schönflies Theorem any simply closed curve C in G divides G into 2 parts, showing that any such C is homologous to a sum of boundary curves, up to global sign. That is, we have half of the following lemma.

LEMMA 5.1. *A homology class $\gamma = \sum \varepsilon_i [B_i] \in H_1(G)$ is represented by a simple closed curve if and only if $\varepsilon_i \in \{0, +1\}$ for all i, or $\varepsilon_i \in \{0, -1\}$ for all i.*

Proof. It remains to show that a given $\gamma = \sum \varepsilon_i [B_i] \in H_1(G)$, with $\varepsilon_i \in \{0, 1\}$ is represented by a simple closed curve. One can organize this process by choosing a tree in G meeting only the boundary curves B_i with coefficient $\varepsilon_i = 1$, and then only in one point for each such B_i. The desired simple closed curve can be chosen to be the interior boundary of a small regular neighborhood of the union of the tree and the boundary curves it meets.

Note, for example, that $[B_1] + [B_2]$ is represented by a simple closed curve, while $[B_1] - [B_2]$ and $[B_1] + 2[B_2]$ are not.

We call a homology class, as in the statement of Lemma 5.1 a *basic* class. Notice that if γ is basic, then so is $-\gamma$. Notice also that a nonzero basic class has a unique representation with all nonnegative coefficients. There are $2^m - 1$ nonzero basic classes, then, that we want to consider.

We now consider a family of homology classes $\gamma_1, \dots, \gamma_k \in H_1(G)$ and ask when they can be represented by pairwise disjoint simple closed curves in G. Using the above lemma together with the fundamental defining relation for the homology of G we may as well assume that each

$$\gamma_i = \sum \varepsilon_{ij} [B_j]$$

where each $\varepsilon_{ij} \in \{0, 1\}$. Now all intersection numbers in G necessarily vanish, so there is no analogue of the Intersection Condition from Theorem 1.

If $\alpha = \sum \varepsilon_i [B_i] \in H_1(G)$ then we define the *partition* of α, denoted part $\alpha = \{C, D\}$, to be the partition of the set $\{[B_i] : i = 1, \dots, n\}$ consisting of the set C of $[B_i]$ that have nonzero coefficients ε_i and its complement D. Note that since the representation of α as such a linear combination is not unique, it is necessary to include discussion of the complementary sum. Note also that α and $-\alpha$ have the same partitions.

PROPOSITION 5.2. *Two basic classes $\alpha_1, \alpha_2 \in H_1(G)$, with corresponding partitions* part $\alpha_i = \{C_i, D_i\}$, *are represented by two disjoint simple closed curves in G if and only if α_1 and α_2 are individually represented by simple closed curves and $C_1 \subset C_2$ or $C_1 \subset D_2$.*

Proof sketch. The point is that the tree used to determine a simple closed curve A_1 for α_1 does not separate G. Therefore, if the support of α_2, or its complement, is disjoint from the support of α_1, then one can find a tree for α_2 in the complement of the tree for α_1 and the boundary curves it touches.

The proof of Proposition 5.2, extends inductively to prove the following result.

PROPOSITION 5.3. *A set of homology classes $S = \{\alpha_1, \ldots, \alpha_k\} \subset H_1(G)$, with corresponding partitions* part$(\alpha_i) = \{C_i, D_i\}$, *is represented by a corresponding set of pairwise disjoint simple closed curves in G if and only if each α_i is individually represented by a simple closed curve and for each i,j $C_i \subset C_j$ or $C_i \subset D_j$.*

COROLLARY 5.4. *A set S of homology classes in $H_1(G)$ is represented by pairwise disjoint simple closed curves in G if and only if any two elements of S are represented by disjoint simple closed curves in G.*

The analogue of the preceding result will be seen to fail for closed surfaces.

COROLLARY 5.5. *A set S of pairwise distinct homology classes in $H_1(G)$ that is represented by pairwise disjoint simple closed curves in G has at most $2m-3$ elements, including the boundary curves.*

Proof. It suffices to assume that S contains no classes homologous to boundary curves and to show that card $S \leq m-3$. Let $k =$ card S and let A denote the union of a set of disjoint simple closed curves in G representing the elements of S. The realization of each element of S divides G into 2 parts. The k elements then divide G into $k+1$ parts X_ℓ. Since the classes in S are not parallel to boundary classes, the components X_ℓ of G cut open along the simple closed curves representing the elements of S all have negative Euler characteristic. Therefore $2-m=\chi(G)=\sum_\ell \chi(X_\ell)\leq (k+1)(-1)$. It follows that $k\leq m-3$, as required.

In general there are many apparently different ways to realize realizable homology classes. But up to homeomorphism we have the following uniqueness result.

THEOREM 5.6. *Suppose that a set of homology classes $S=\{\alpha_1,\dots,\alpha_k\}\subset H_1(G)$ is represented by two different families $A_1,A_2,\dots,A_k$ and $A_1',A_2',\dots,A_k'$ of pairwise disjoint simple closed curves in the planar surface G. Then there is a homeomorphism $f:G\to G$ inducing the identity on homology such that $f(A_j)=A_j'$ for $j=1,\dots,k$.*

We note that the analogue of Theorem 5.6 for closed surfaces is false. We also note that this result shows that in the process of realizing a realizable family of homology classes one-by-one, one cannot get "stuck", which can in fact happen in the case of closed surfaces.

Proof of Theorem 5.6. The overall argument will be by induction on the the number k of homology classes in question. We can assume that G has at least 3 boundary curves. Then any homeomorphism inducing the identity on homology will map each boundary component into itself. It follows that we can assume that the set S of homology classes contains no boundary classes. First consider the case $k=1$ of just one non-boundary class α_1 and two different simple closed curves A_1 and A_1' realizing it. Note that the same boundary curves appear on corresponding sides of A_1 and of A_1'. It follows easily from the Schönflies Theorem that there is a homeomorphism moving A_1 onto A_1' and preserving the corresponding sides. One can then arrange that this homeomorphism induce the identity on the boundary by composing with a homeomorphism that appropriately permutes the boundary curves while leaving A_1' invariant. To argue this in a little more detail, let $\overline{G}$ denote the 2-sphere obtained by collapsing all the boundary curves to single points. The the usual Schönflies Theorem shows that there is a homeomorphism of $\overline{G}$ mapping A_1 onto A_1'. By composing with a homeomorphism that exchanges the two sides of A_1' if necessary, we can assume that this homeomorphism maps the complementary domains of A_1 to the corresponding complementary domains of A_1'. Then homogeneity shows that one can further arrange that this homeomorphism can be arranged to map each ideal point to itself. One can then "blow up" the ideal points to the original boundary curves.

Now, proceeding inductively, consider the case of $k > 1$ homology classes. One of these homology classes, say α_k, has a minimal partition C_k, D_k, in the sense that C_k contains no other C_j or D_j for j$< k$. By the preceding argument we may assume that $A_k = A'_k$. One side of A_k contains no other simple closed curves A_j or A'_j. Excise this side to obtain a new planar surface H containing the remaining simple closed curves. By induction there is a homeomorphism h of H moving A_j onto A'_j for $1 \le j \le k-1$. and mapping each boundary curve into itself. We can then reinsert the excised domain to complete the argument.

The results of this section, with the exception of Theorem 5.6 above, hold *mutatis mutandi* for compact non-planar surfaces G with boundary, provided one only considers homology classes given as linear combinations of the classes represented by the boundary curves. Each such simple closed curve in the interior of G would separate G. Uniqueness, however, is obstructed by needing to know the genus of each complementary domain.

6. Sufficiency in Theorem 3

Let $S \subset H_1(F)$ denote a finite set of distinct homology classes satisfying the Intersection Condition, the Summand Condition, and the Size Condition of Theorem 1, which we wish to represent by pairwise disjoint simple closed curves. Suppose that the linear span of S has rank n and extract from S n elements $\alpha_1, \ldots, \alpha_n$ that form a basis for this span. Now each element γ_i in the remaining subset T of S can be expressed as a linear combination

$$\gamma_i = \sum_j \varepsilon_{ij} \, \alpha_j \, .$$

We refer to the γ_i as "composite classes."

LEMMA 6.1. *Each coefficient ε_{ij} in the linear combination $\gamma_i = \sum_j \varepsilon_{ij} \, \alpha_j$ is 1, -1, or 0.*

Proof. To see this, consider the span of the set consisting of any one γ_i together with all α_k, $k \neq j$. Elementary change of basis operations show that this span is the same as the span of $\varepsilon_{ij} \, \alpha_j$ and the α_k, $k \neq j$. By the Summand Condition, this span must be a summand, and it therefore follows that $\varepsilon_{ij} = \pm 1$ or 0.