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homology class in F, up to sign. It follows, therefore, from the classification of surfaces, that each  $X_{\ell}$  has Euler characteristic  $\leq -1$ . Therefore, when n > 1 and n = g,

$$\chi(F) = \chi(\bar{F}) = 2 - 2n = \sum \chi(X_{\ell}) \le (\text{card } S - n + 1)(-1)$$

or equivalently card  $S \leq 3n-3$ , as required.

It remains to consider the case when g > n > 1. In this case, we first proceed as before, cutting open along the  $A_i$ , obtaining a connected surface  $\widehat{F}$  of genus g-n and with 2n boundary curves, containing the m curves  $C_j$ , each of which is homologous to a sum of boundary curves in  $\widehat{F}$ . Now each of the  $C_j$  separates  $\widehat{F}$ , and we may further cut open along the  $C_j$ , obtaining a surface with m+1 components and total genus g-n. It follows that there are additional pairwise disjoint simple closed curves  $E_k$ ,  $k=1,\ldots,g-n$ , in  $\widehat{F}$ , reducing  $\widehat{F}$  to a planar surface of genus 0 when we cut open along the  $E_k$  and cap off the resulting 2(g-n) boundary curves with disks. Call this latter surface  $\overline{F}$ , topologically a 2-sphere with 2n holes. Now the  $C_j$  separate  $\overline{F}$  into m+1= card S-n+1 planar components  $X_\ell$ . As before, each  $X_\ell$  has Euler characteristic  $\leq -1$ . Therefore, again,

$$\chi(\bar{F}) = 2 - 2n = \sum \chi(X_{\ell}) \le (\text{card } S - n + 1)(-1)$$

or equivalently card  $S \leq 3n - 3$ , as required.

# 3. Sufficiency for a single homology class

Here we collect some basic information about the embedding of a single simple closed curve in a surface, and offer an alternative, elementary proof of Theorem 2 for the well-known case of a single homology class.

LEMMA 3.1. A nonzero homology class  $\alpha \in H_1(F)$  is primitive if and only if there exists  $\gamma \in H_1(F)$  such that  $\gamma \cdot \alpha = 1$ .

*Proof.* A nonzero element of a finitely generated free abelian group is primitive if and only if it is part of a basis if and only if there is a **Z**-valued homomorphism that takes the value 1 on it. Recall that taking intersection numbers of 1-cycles defines a skew symmetric bilinear form on  $H_1(F)$ . The content of Poincaré Duality in this situation is that this bilinear form is nonsingular, that is, the adjoint homomorphism  $H_1(F) \to \operatorname{Hom}(H_1(F), \mathbf{Z})$  is an isomorphism. The lemma then follows.

LEMMA 3.2. Any homology class  $\alpha \in H_1(F)$  can be represented by an immersed, oriented closed curve on F and also by an embedded, oriented 1-submanifold.

*Proof.* The Hurewicz homomorphism  $\pi_1(F) \to H_1(F)$  is onto. Compare W. Massey [1980], Chapter III, Section 7, for example. Any map  $S^1 \to F$  can be approximated by an immersion, with only isolated double points. One can surger any double points, that is, one can replace any pair of small oriented arcs having a single transverse intersection with a pair of parallel oriented arcs with the same end points and lying within a regular neighborhood of the intersecting arcs. In this way one creates a disjoint union of oriented simple closed curves representing the same homology class.

PROPOSITION 3.3. A homology class  $\alpha$  in  $H_1(F)$  can be represented by a simple closed curve on F if and only if  $\alpha$  is primitive.

*Proof.* We sketch a 2-dimensional version of the argument of Bennequin [1977]. If a simple closed curve represents a nonzero homology class, then it is nonseparating. It follows that there is a simple closed curve that meets it transversely in a single point. This implies indivisibility, by the homology invariance of intersection numbers.

For the converse, we may assume that  $\alpha$  is nonzero. We begin by representing  $\alpha$  by a disjoint union A of oriented simple closed curves, as in Lemma 3.2. We shall assume that A contains the smallest possible number of components and show that this number can be reduced unless it is 1 or it is equal to the divisibility of  $\alpha$ .

Cut open F along A-that is, remove the interior of a small tubular neighborhood of A. The boundary of the cut open surface  $\widehat{F}$  consists of two copies  $A_i^+$  and  $A_i^-$  of each component  $A_i$  of A, each of which we orient as the boundary of the orientable surface  $\widehat{F}$ . The positive components  $A_i^+$  have the same orientation as  $A_i$ , while the negative components  $A_i^-$  have the opposite orientation.

If some component R of  $\widehat{F}$  contains in its boundary two positive curves  $A_i^+$  and  $A_j^+$  (or two negative curves), then they can be banded together in an orientable way using a band in R. That is, one chooses an embedded arc  $\delta$  in R meeting  $A_i^+$  and  $A_j^+$  in its two end points only. One then replaces  $A_i$  and  $A_j$  with the single simple closed curve obtained by removing small arcs in  $A_i$  and  $A_j$  containing the end points of  $\delta$  and inserting in their place two embedded arcs parallel to  $\delta$ . This would reduce the number of components of A. If some component R has boundary just  $A_i^+$  and  $A_i^-$  for some  $A_i$ ,

then we can conclude that A is connected and we are done. If some R has more than two boundary components, then it contains two positive curves or two negative curves, and we can proceed as above to reduce the number of components of A.

It remains to consider the case where each component  $R_k$  of  $\widehat{F}$  has exactly two boundary components of the form  $A_i^+$  and  $A_j^-$ , where  $A_i$  and  $A_j$  are distinct components of A. In this case we conclude that we can arrange the components of A in a sequence  $A_1, A_2, \ldots, A_n$ , so that  $A_1$  is homologous to  $A_2, A_2$  is homologous to  $A_3, \ldots, A_n$  is homologous to  $A_1$ . In this case, then, the number n of components is exactly the divisibility of  $\alpha$ .

## 4. SUFFICIENCY FOR INDEPENDENT HOMOLOGY CLASSES

In this section we complete the proof of Theorem 2, dealing with the case of a set of homology classes consisting of independent elements.

LEMMA 4.1. Let F be a closed orientable surface and let  $\alpha_1, \ldots, \alpha_n \in H_1(F)$  be independent homology classes that span a summand of  $H_1(F)$  on which the intersection pairing of F vanishes. Then there exists  $\gamma \in H_1(F)$  such that  $\gamma \cdot \alpha_n = 1$  and  $\gamma \cdot \alpha_i = 0$  for i < n.

Proof. This is a consequence of Poincaré Duality.

PROPOSITION 4.2. Let F be a closed orientable surface and let  $\alpha_1, \ldots, \alpha_n \in H_1(F)$  be independent homology classes that span a summand of  $H_1(F)$  on which the intersection pairing of F vanishes. Then there exist pairwise disjoint simple closed curves  $A_1, \ldots, A_n$  in F representing the homology classes  $\alpha_1, \ldots, \alpha_n$ .

*Proof.* The proof will proceed by induction on n. The case n=1 is given by Proposition 3.3.

Now inductively consider the case of n>1 homology classes. By Proposition 3.3 we can find a simple closed curve  $A_n$  in F representing  $\alpha_n$ . We claim that there is a simple closed curve  $B_n$  in F representing a homology class  $\beta_n$  such that  $B_n$  meets  $A_n$  in exactly one point and such that  $[B_n] \cdot \alpha_i = 0$  for i < n. By Lemma 4.1 there is a homology class  $\gamma_n \in H_1(F)$  such that  $\alpha_i \cdot \gamma_n = \delta_{i,n}$ . We begin by representing  $\gamma_n$  by a simple closed curve B transverse to  $A_n$ . By tubing together neighboring pairs of intersection of B with  $A_n$  of opposite sign we can transform B into a disjoint union B'