

<b>Zeitschrift:</b>	L'Enseignement Mathématique
<b>Herausgeber:</b>	Commission Internationale de l'Enseignement Mathématique
<b>Band:</b>	42 (1996)
<b>Heft:</b>	3-4: L'ENSEIGNEMENT MATHÉMATIQUE
<b>Artikel:</b>	SYSTEMS OF CURVES ON A CLOSED ORIENTABLE SURFACE
<b>Autor:</b>	EDMONDS, Allan L.
<b>Kapitel:</b>	2. Necessary conditions
<b>DOI:</b>	<a href="https://doi.org/10.5169/seals-87881">https://doi.org/10.5169/seals-87881</a>

#### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

#### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

#### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 19.01.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

We conclude by proving an analogue of this for homologically nontrivial and distinct curves.

**THEOREM 7.** *Let  $F$  be a closed orientable surface of genus  $g \geq 2$ , and let  $S \subset H_1(F)$  be a set of pairwise distinct nonzero homology classes represented by a corresponding family of pairwise disjoint simple closed curves in  $F$ . Then this family of simple closed curves can be extended to a family of  $3g - 3$  pairwise disjoint simple closed curves in  $F$  representing nontrivial, pairwise distinct homology classes in  $H_1(F)$ .*

Here is a summary of the contents of the rest of the paper: Section 2 contains the proof of Theorem 1 deriving the fundamental necessary conditions. Sections 3 and 4 deal with the cases of one homology class and with independent homology classes, and provide two proofs of Theorem 2. In Section 5 we give an analysis of simple closed curves on a planar surface. In Section 6 there is the proof of the main positive realizability statement, Theorem 3, ending with a discussion of Theorem 6. In Section 7 we present several examples that illustrate some of the subtleties involved in finding a more complete and definitive result than that given here, including the nonrealizability result stated as Theorem 4. Finally in Section 8 we give the proofs of Theorem 5 and 7.

The author acknowledges helpful conversations with Chuck Livingston, especially in the early stages of this work, useful comments from Michel Kervaire, and the hospitality of the Max Planck Institut für Mathematik in Bonn, where some of the work was completed.

## 2. NECESSARY CONDITIONS

It is quite clear that the Intersection Condition must hold, since the intersection number of two disjoint 1-cycles is necessarily 0. The necessity of the Summand Condition follows immediately from the following lemma.

**LEMMA 2.1.** *Let  $F$  be a closed orientable surface of genus  $g \geq 1$  and let  $S \subset H_1(F)$  be a set of pairwise distinct homology classes represented by a corresponding set of pairwise disjoint simple closed curves in  $F$ . Then the image of  $S$  spans a direct summand of  $H_1(F)$ .*

*Proof.* Let  $A \subset F$  be the union of the simple closed curves representing the elements of  $S$  in  $H_1(F)$ . Consider the long exact homology sequence of the pair  $(F, A)$ .

$$\cdots \rightarrow H_1(A) \rightarrow H_1(F) \rightarrow H_1(F, A) \rightarrow \cdots$$

Now the linear span of  $S$  in  $H_1(F)$  is identified with the image of  $H_1(A)$  in  $H_1(F)$ . But  $H_1(F, A)$  is free (by Poincaré Duality), so we see that the image of  $H_1(A)$  is a direct summand, as required.

The following result gives the Size Condition. The construction described in the proof below will be important, as it describes an effective way to approach the problem of explicitly realizing a system of pairwise disjoint curves.

**LEMMA 2.2.** *Let  $F$  be a closed orientable surface of genus  $g \geq 1$ , let  $S \subset H_1(F)$  be a set of pairwise distinct homology classes represented by a corresponding set of pairwise disjoint simple closed curves in  $F$ , and let  $n = \text{rank span } S$ . Then  $\text{card } S \leq \max\{3n - 3, 1\}$ .*

*Proof.* If  $n = 1$ , then it follows from Lemma 2.1 that  $S$  must consist of a single element, and the desired inequality trivially holds. Henceforth we assume that  $n > 1$ . The proof in this case will amount to cutting the surface up into pieces along the given simple closed curves and using the pieces to calculate the euler characteristic of the surface. It is easy to see that  $g \geq n$ . We will first assume that  $g = n$ . At the end we will indicate how to modify the proof to handle the case  $g > n$ .

Let  $\alpha_1, \dots, \alpha_n \in S$  form a basis for  $\text{span } S$  and let  $A_1, \dots, A_n$  be the corresponding disjoint simple closed curves in  $F$ . Let  $\gamma_1, \dots, \gamma_m \in S$  be the remaining elements of  $S$  and  $C_1, \dots, C_m$  be the corresponding disjoint simple closed curves in  $F$ . Let  $\widehat{F}$  denote the surface  $F$  cut open along the  $A_i$ . Then  $\widehat{F}$  is a connected, orientable surface and has  $2n$  boundary curves and genus  $g - n = 0$ . Note that  $\chi(\widehat{F}) = \chi(F)$  by the sum formula for euler characteristics. In  $\widehat{F}$  each of the  $m = \text{card } S - n$  curves  $C_j$  is homologous to a sum of boundary curves, with multiplicities  $\pm 1$ , since  $C_j$  does not separate  $F$ , but does separate  $\widehat{F}$ . Now  $\bar{F} = \widehat{F} - \cup C_j$  still has genus 0 and consists of  $m + 1$  planar components  $X_\ell$ . Again note that  $\chi(\bar{F}) = \chi(\widehat{F})$ . No  $X_\ell$  can be a disk, since otherwise its boundary curve would be nullhomologous in  $F$ . Similarly, no  $X_\ell$  can be an annulus, since otherwise, the two boundary curves, belonging to the original collection of curves would represent the same

homology class in  $F$ , up to sign. It follows, therefore, from the classification of surfaces, that each  $X_\ell$  has Euler characteristic  $\leq -1$ . Therefore, when  $n > 1$  and  $n = g$ ,

$$\chi(F) = \chi(\bar{F}) = 2 - 2n = \sum \chi(X_\ell) \leq (\text{card } S - n + 1)(-1)$$

or equivalently  $\text{card } S \leq 3n - 3$ , as required.

It remains to consider the case when  $g > n > 1$ . In this case, we first proceed as before, cutting open along the  $A_i$ , obtaining a connected surface  $\hat{F}$  of genus  $g - n$  and with  $2n$  boundary curves, containing the  $m$  curves  $C_j$ , each of which is homologous to a sum of boundary curves in  $\hat{F}$ . Now each of the  $C_j$  separates  $\hat{F}$ , and we may further cut open along the  $C_j$ , obtaining a surface with  $m + 1$  components and total genus  $g - n$ . It follows that there are additional pairwise disjoint simple closed curves  $E_k$ ,  $k = 1, \dots, g - n$ , in  $\hat{F}$ , reducing  $\hat{F}$  to a planar surface of genus 0 when we cut open along the  $E_k$  and cap off the resulting  $2(g - n)$  boundary curves with disks. Call this latter surface  $\bar{F}$ , topologically a 2-sphere with  $2n$  holes. Now the  $C_j$  separate  $\bar{F}$  into  $m + 1 = \text{card } S - n + 1$  planar components  $X_\ell$ . As before, each  $X_\ell$  has Euler characteristic  $\leq -1$ . Therefore, again,

$$\chi(\bar{F}) = 2 - 2n = \sum \chi(X_\ell) \leq (\text{card } S - n + 1)(-1)$$

or equivalently  $\text{card } S \leq 3n - 3$ , as required.

### 3. SUFFICIENCY FOR A SINGLE HOMOLOGY CLASS

Here we collect some basic information about the embedding of a single simple closed curve in a surface, and offer an alternative, elementary proof of Theorem 2 for the well-known case of a single homology class.

**LEMMA 3.1.** *A nonzero homology class  $\alpha \in H_1(F)$  is primitive if and only if there exists  $\gamma \in H_1(F)$  such that  $\gamma \cdot \alpha = 1$ .*

*Proof.* A nonzero element of a finitely generated free abelian group is primitive if and only if it is part of a basis if and only if there is a  $\mathbf{Z}$ -valued homomorphism that takes the value 1 on it. Recall that taking intersection numbers of 1-cycles defines a skew symmetric bilinear form on  $H_1(F)$ . The content of Poincaré Duality in this situation is that this bilinear form is nonsingular, that is, the adjoint homomorphism  $H_1(F) \rightarrow \text{Hom}(H_1(F), \mathbf{Z})$  is an isomorphism. The lemma then follows.