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SYSTEMS OF CURVES ON A CLOSED ORIENTABLE SURFACE

by Allan L. EDMONDS

1. INTRODUCTION

It is well-known that a nontrivial one-dimensional homology class on a closed orientable surface F is represented by a simple closed curve in F if and only if it is *primitive*, i.e., indivisible. See Myerson [1976], Bennequin [1977], and Meeks-Patrusky [1978]. (There is also a partial result in Kaneko-Aoki-Kobayashi [1963].) Here we study the more general question of when a *collection* of pairwise distinct homology classes is represented by a set of corresponding pairwise disjoint simple closed curves. We first introduce the following necessary conditions.

THEOREM 1. *Let F be a closed orientable surface and let $S \subset H_1(F)$ be a set of pairwise distinct nonzero homology classes. If S is represented by a corresponding set of pairwise disjoint simple closed curves in F then the following three conditions are satisfied:*

1. INTERSECTION CONDITION. *The intersection pairing of F vanishes on S.*
2. SUMMAND CONDITION. *Every subset T of S spans a direct summand* span T *of $H_1(F)$.*
3. SIZE CONDITION. *For every subset T of S of more than one element* card $T \leq 3$ rank span $T - 3$.

Here we say that two homology classes α and β are *distinct* if $\alpha \neq \beta$ and $\alpha \neq -\beta$. Although linear algebraic in nature, the Summand Condition

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and Size Condition can be a little tricky to check in specific cases. None of these three conditions follows from the others in general.

We then investigate the attractive conjecture that these natural necessary conditions are in fact sufficient. In this direction we begin with the case of independent homology classes.

THEOREM 2. *Let F be a closed orientable surface and let $S \subset H_1(F)$ be a set of pairwise distinct, independent, and indivisible homology classes. Then S is represented by a corresponding set of pairwise disjoint simple closed curves in F if and only if the Intersection Condition holds and S spans a summand of $H_1(F)$.*

Here is a simple interpretation of the Theorem 2 in the case of just two homology classes, which is the case with which the present investigation started.

COROLLARY. *Let F be a closed orientable surface of genus $g \geq 2$ and let $\alpha_1, \alpha_2 \in H_1(F)$ be two distinct homology classes. Then α_1 and α_2 are represented by disjoint simple closed curves in F if and only if α_1 and α_2 are indivisible, $\alpha_1 \cdot \alpha_2 = 0$, and α_2 is indivisible in $H_1(F)/(\alpha_1)$.)*

Here are some simple interpretations of these basic results. Let

$$\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g$$

denote a standard symplectic basis for the homology of F. In particular, this means that these homology classes are represented by simple closed curves

$$A_1, B_1, A_2, B_2, \ldots, A_g, B_g$$

in F such that the A_i are pairwise disjoint, the B_j are pairwise disjoint, and if $A_i \cap B_j \neq \varnothing$, then $i = j$ and $A_i \cap B_j =$ a single point of transverse intersection. Then the corollary says that α_1 and $2\alpha_1 + \alpha_2$ are represented by disjoint simple closed curves, as one can easily check by hand, drawing suitable pictures. On the other hand, α_1 and $\alpha_1 + 2\alpha_2$ are not so represented. Note further that one can represent the three classes α_1, α_2, and $\alpha_1 + \alpha_2$ by disjoint simple closed curves, by explicitly drawing the curves. By Theorem 1, no more than 3 such classes can be so represented on a surface of genus 2. On a surface of genus 3 one can easily construct 6 pairwise disjoint simple closed curves representing distinct homology classes.

Again, Theorem 1 implies that on a surface of genus 3 one cannot realize 7 distinct classes this way. And so on. More subtle examples will be discussed later.

To consider the more general cases of not-necessarily independent homology classes we introduce the following terminology. Define the *rank* of S, rank S, to be the rank of the integral span of S in $H_1(F)$. And define the *excess* of S, excess S, to be card $S -$ rank S. Through a fairly painstaking and increasingly subtle analysis we are able to prove sufficiency of the conditions above when either the excess or rank is not too big.

THEOREM 3. *Let F be a closed orientable surface and let $S \subset H_1(F)$ be a set of pairwise distinct nonzero homology classes satisfying the Intersection Condition, Summand Condition, and Size Condition. Then S is represented by corresponding pairwise disjoint simple closed curves in F provided that either* excess $S \leq 3$ *or* rank $S \leq 4$.

The increasing difficulties encountered while attempting to extend the result of Theorem 3 eventually led to a family of counterexamples as described in the following result.

THEOREM 4. *Let F be a closed orientable surface of genus at least 5. Then there is a family $S \subset H_1(F)$ of 9 pairwise distinct nonzero homology classes satisfying the Intersection Condition, Summand Condition, and Size Condition and having excess 4 and rank 5 that is not representable by a corresponding family of pairwise disjoint simple closed curves in F.*

In particular Theorem 4 destroys all sorts of natural inductive approaches to proving realizability of families of homology classes by pairwise disjoint simple closed curves. We include in Section 7 of this paper some additional examples that illustrate the difficulties in proving realizability, including an example of realizable homology classes such that there is a realization of all but one of them that cannot be extended to a realization of the whole family.

A natural hope would be that perhaps the necessary conditions in Theorem 1 are at least sufficient after suitable stabilization or connected sum with a suitable number of tori. But this turns out not to be the case.

THEOREM 5. *Let F be a closed orientable surface and let $S \subset H_1(F)$ be a set of pairwise distinct nonzero homology classes satisfying the Intersection Condition, Summand Condition, and Size Condition. Suppose that the corresponding set of homology classes in $F\#F_1$ is represented by corresponding pairwise disjoint simple closed curves in $F\#F_1$ for some closed orientable surface F_1. Then S is represented by corresponding pairwise disjoint simple closed curves in F.*

The analysis in Theorems 3 and 4 is based upon realizing a maximal subcollection of independent classes by simple closed curves and then cutting open the given surface to form a surface with boundary. This surface can effectively be viewed as being planar. Then the problem of realizing any remaining classes is reduced to lifting the classes to homology classes in the punctured surface (which are not uniquely defined) and realizing them there. Thus we also include a preliminary step in which we give a complete analysis of the corresponding but much easier problem of realizing a family of homology classes in a compact planar surface by pairwise disjoint simple closed curves. One attractive statement in this context is that a family of homology classes in a planar surface is realizable by a corresponding family of pairwise disjoint simple closed curves if and only if each subcollection of two elements is so realizable. The analogue of this statement for closed surfaces is false.

An important consequence of this analysis of planar surfaces is the following result.

THEOREM 6. *Let F be a closed orientable surface and let $S \subset H_1(F)$ be a set of pairwise distinct nonzero homology classes satisfying the Intersection Condition, Summand Condition, and Size Condition. Then there is a finite (but "exponential") algorithm for deciding whether S can be represented by corresponding pairwise disjoint simple closed curves in F.*

We have written computer programs in Maple that in principle can carry out such an algorithm. Unfortunately, at the time of this writing the first interesting cases are too large for the current version of the programs to be effective. (The program did assist in enumerating the cases where rank $S = 4$ that were analyzed in Theorem 3.)

It is elementary to see that any family of homotopically nontrivial and nonparallel pairwise disjoint simple closed curves in a surface of genus g can be extended to a maximal family of $3g-3$ such simple closed curves.

We conclude by proving an analogue of this for homologically nontrivial and distinct curves.

THEOREM 7. *Let F be a closed orientable surface of genus $g \geq 2$, and let $S \subset H_1(F)$ be a set of pairwise distinct nonzero homology classes represented by a corresponding family of pairwise disjoint simple closed curves in F. Then this family of simple closed curves can be extended to a family of $3g-3$ pairwise disjoint simple closed curves in F representing nontrivial, pairwise distinct homology classes in $H_1(F)$.*

Here is a summary of the contents of the rest of the paper: Section 2 contains the proof of Theorem 1 deriving the fundamental necessary conditions. Sections 3 and 4 deal with the cases of one homology class and with independent homology classes, and provide two proofs of Theorem 2. In Section 5 we give an analysis of simple closed curves on a planar surface. In Section 6 there is the proof of the main positive realizability statement, Theorem 3, ending with a discussion of Theorem 6. In Section 7 we present several examples that illustrate some of the subtleties involved in finding a more complete and definitive result than that given here, including the nonrealizability result stated as Theorem 4. Finally in Section 8 we give the proofs of Theorem 5 and 7.

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2. NECESSARY CONDITIONS

It is quite clear that the Intersection Condition must hold, since the intersection number of two disjoint 1-cycles is necessarily 0. The necessity of the Summand Condition follows immediately from the following lemma.

LEMMA 2.1. *Let F be a closed orientable surface of genus $g \geq 1$ and let $S \subset H_1(F)$ be a set of pairwise distinct homology classes represented by a corresponding set of pairwise disjoint simple closed curves in F. Then the image of S spans a direct summand of $H_1(F)$.*