1. Introduction

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BARBIER'S THEOREM FOR THE SPHERE AND THE HYPERBOLIC PLANE

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1. INTRODUCTION

Curves of constant width are a class of plane curves with surprising properties, not the least being the very existence of such curves which are not circles (see ch. 25 of [RT] for an elementary introduction). Also remarkable is the fact that all curves of the same constant width have the same perimeter: this is the content of Barbier's theorem (whose original proof, using probabilistic methods, appears in [B] and is reproduced in [C], pp. 161-163). Here we investigate how Barbier's theorem generalizes to the complete, simply connected surfaces of constant curvature K, which we denote by S_K (thus S_K is the sphere of radius $\frac{1}{\sqrt{K}}$ if $K > 0$; the euclidean plane if $K = 0$; and the hyperbolic plane with curvature K if $K < 0$). The formulas in our main Theorem B are originally due to Blaschke (for $K > 0$) and Santaló (for $K < 0$), but it seems worthwhile to bring them together using a unified differential geometric approach.

Curves of constant width are defined as follows. Let γ be a convex closed Jordan curve on the euclidean plane, and consider the strips Ω bounded by two parallel lines r_0, r_1 such that $\gamma \subseteq \Omega$ and both r_0, r_1 touch γ at some point. We call such sets Ω *enveloping strips* of γ. There is exactly one enveloping strip of γ which is parallel to any given direction. If all enveloping strips of γ have the same width $\mathcal{W}$ then we say γ has *constant width* $\mathcal{W}$.

The following lemma gives an alternative characterization of these curves. For a curve γ and a point p on it, we let $\mathcal{D}(p) = \max\{|p - q| : q \in \gamma\}$.

LEMMA A. *A simple closed curve γ is of constant width $\mathcal{W}$ if and only if $\mathcal{D}(p) = \mathcal{W}$ for all $p \in \gamma$.*

This lemma allows us to define curves of constant width in metric spaces other than the euclidean plane, and in particular we consider them in regular (differentiable) surfaces endowed with a riemannian metric and associated intrinsic distance. We prove the following result extending Barbier's theorem to S_K.

THEOREM B ([Bl], [S]). *If a curve of constant width $\mathcal{W}$ in S_K has perimeter $\mathcal{L}$ and bounds a region of area $\mathcal{A}$ then*

$$\mathcal{L} = H(K, \mathcal{W}) \{2\pi - K\mathcal{A}\},$$

where $H(K, \mathcal{W})$ is given by

$$\frac{1}{\sqrt{K}} \tan\left(\frac{\sqrt{K}\,\mathcal{W}}{2}\right) \text{ if } K > 0; \frac{\mathcal{W}}{2} \text{ if } K = 0; \frac{1}{\sqrt{-K}} \tanh\left(\frac{\sqrt{-K}\,\mathcal{W}}{2}\right) \text{ if } K < 0.$$

There exists a generalization for S_K of the well-known isoperimetric inequality for the euclidean plane. It states that if a curve γ in S_K of perimeter $\mathcal{L}$ bounds a region of area $\mathcal{A}$ then

$$\mathcal{L}^2 \geq 4\pi\mathcal{A} - K\mathcal{A}^2, \tag{1}$$

and that the equality holds if and and only if γ is a (geodesic) circle (see [O] for a proof). From (1) and the above theorem we obtain the following corollary :

COROLLARY C. *If $K > 0$ (respectively $K < 0$) then, of all the curves γ in S_K with the same constant width $\mathcal{W}$, the circle has the least (resp. the greatest) perimeter. More precisely, we have*

$$\mathcal{L} \geq \frac{2\pi}{\sqrt{K}} \sin\left(\frac{\sqrt{K}\,\mathcal{W}}{2}\right) \qquad \left[\textit{resp. } \mathcal{L} \leq \frac{2\pi}{\sqrt{-K}} \sinh\left(\frac{\sqrt{-K}\,\mathcal{W}}{2}\right)\right],$$

and equality holds if and only if γ is a circle.

From Theorem B we see that, for fixed $\mathcal{W}$ and $K < 0$, curves of longer perimeter enclose larger areas, whereas for $K > 0$ larger areas correspond to shorter perimeters. Thus Corollary C can be expressed by saying that, in all cases, the curve of a given width enclosing the largest area is the circle (for $K = 0$ this follows immediately from combining Barbier's theorem with the isoperimetric inequality).

If γ is a curve of constant width $\mathcal{W}$ in S_K, we say that $p, \tilde{p} \in \gamma$ are *antipodal points* if the (intrinsic) distance between them is $\mathcal{W}$, which is to say that they realize the diameter of γ. We prove a result that was already known in the case of the euclidean plane (see [HS]):

THEOREM D. *If γ is a curve of constant width $\mathcal{W}$ in S_K such that every pair of antipodal points divides γ into two arcs of equal length (and, in the case of the sphere, if $\mathcal{W} < \frac{\pi}{\sqrt{K}}$) then γ is a circle.*

We must emphasize that, except for Lemma A, proofs are only given for *regular curves*, which for us means that they have no corners and the natural parametrization by arc-length is C^∞ (or just C^k for big enough k). By the expedient of using parallel curves, as explained in the next section, we can extend our results to curves consisting of regular pieces and a *finite* number of corners *(piecewise regular curves)*, but further extension does not seem possible using our methods.

The remainder of this article is organized as follows. In the next section we discuss curves of constant width in the familiar setting of the euclidean plane, and prove Lemma A and Barbier's theorem. Our proof of Barbier's theorem is similar to that in section 1.13 of [St], but we choose to present it here since the proof we give for S_K $(K \neq 0)$ is an elaboration of our proof for S_0.

In §3 we consider general oriented surfaces and construct systems of *geodesic parallel coordinates* suitable for dealing with our curves, proving a number of technical results about these coordinates, and also proving Theorem D. In the last section all pieces are put together to give the proofs of Theorem B and Corollary C.

2. CURVES OF CONSTANT WIDTH IN THE EUCLIDEAN PLANE

We now review some background on convex curves; the basic reference here is [E]. Given a closed curve $\gamma \subseteq \mathbf{R}^2$, a straight line r is called a *supporting line* of γ if r touches γ at some point and γ is entirely contained in one of the closed half-planes bounded by r. One possible characterization of convex curves is the following: γ is convex if and only if through every point of γ there passes a supporting line of γ. If some boundary line of an