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**Autor:** HEITSCH, James L.  
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$w_\alpha^L(z)$  on  $N_\alpha \cap L$  which represents the cohomology class  $\widehat{A}(N_\alpha \cap L)B(N_\alpha \cap L, z)$ . Then  $w_\alpha^L(z)$  is the form  $a_\alpha^L$  given in the foliation Lefschetz theorem for  $z$  acting on the leafwise Spin complex, and it defines a smooth form  $w_\alpha(z)$  on  $N_\alpha$ . Thus for  $z \in S^1$ ,  $z$  not a root of unity, we have

$$L(z) = \int_N w(z) = \sum_\alpha \int_{N_\alpha} w_\alpha(z).$$

Now notice that the right side of this equation defines a function  $A(F, z)$  on the complex plane with values in the Haefliger forms of  $F$ . Also note that  $A(F, z)$  has poles only at roots of unity and no pole at  $z = \infty$ , since  $w_\alpha(z)$  has poles only at roots of unity and no pole at  $z = \infty$ . Because of the factor of  $(z^d)^{1/2}$ ,  $A(F, 0) = 0$ . For  $z \in S^1$ ,  $z$  not a root of unity,  $A(F, z) = L(z)$ . But  $L(z)$  is defined for all  $z \in S^1$  and by Theorem 5.5 it is continuous on  $S^1$ . Thus  $A(F, z)$  has no poles at all. Since it is analytic and bounded, it is constant and hence is identically zero. Therefore  $L(z) = 0$  for all  $z \in S^1$ , but  $L(1) = \widehat{A}(F)$  so we are done.

The compactness of  $G$  is essential, as in [HL2], we give an example of an infinite discrete group acting by leaf preserving isometries on a compact oriented foliated manifold  $M, F$  and  $G$  preserves a Spin structure on  $F$ . The foliation  $F$  admits an invariant transverse measure which defines a map from the Haefliger zero forms of  $F$  to  $\mathbf{C}$ . The image of  $\widehat{A}(F)$  under this map is non-zero, so  $\widehat{A}(F) \neq 0$ .

## 6. THE RIGIDITY THEOREM OF WITTEN

In 1986, Witten [W] predicted rigidity theorems for the indices of certain elliptic operators on manifolds with  $S^1$  actions. The genesis for Witten's conjecture was his study of the Dirac operator on the free loop space  $\mathcal{L}M$  (an infinite dimensional manifold) of a Spin manifold  $M$ .  $\mathcal{L}M$  admits a natural  $S^1$  action whose fixed point set is diffeomorphic to  $M$ . The sequences of bundles  $R(q)$  and  $R'(q)$  described below were derived from the normal bundle of  $M$  in  $\mathcal{L}M$  and from the formal analogue on  $\mathcal{L}M$  of the fixed point formula for the Dirac operator in the finite dimensional case.

Let  $D : C^\infty(E_1) \rightarrow C^\infty(E_2)$  be an elliptic operator on a compact manifold  $M$  and suppose  $M$  admits an  $S^1$  action preserving  $D$ . Then as noted above,  $\text{Index}(D)$  is a virtual  $S^1$  module and has a decomposition into a finite sum of irreducible complex one dimensional representations

$$\text{Index}(D) = \sum a_m L^m$$

where  $z \in S^1$  acts on  $L^m$  by multiplication by  $z^m$ .  $D$  is called *rigid* if all the  $a_m$  for  $m \neq 0$  are zero, i.e. if the representation  $L^m$ ,  $m \neq 0$  occurs in kernel  $D$  with multiplicity  $a$  then it occurs in cokernel  $D$  with the same multiplicity  $a$ .

Denote by  $S^k(T)$  and  $\lambda^k(T)$  the  $k$ th symmetric and exterior powers of  $T = TM$  and set

$$S_a(T) = \sum_{k=0}^{\infty} a^k S^k(T)$$

$$\lambda_a(T) = \sum_{k=0}^{\infty} a^k \lambda^k(T).$$

Let  $R_n$  and  $R'_n$  be the sequences of bundles defined by the formal power series

$$R(q) = \sum_{n=0}^{\infty} q^n R_n = \bigotimes_{\ell=1}^{\infty} \lambda_{q^\ell}(T) \bigotimes_{m=1}^{\infty} S_{q^m}(T)$$

$$R'(q) = \sum_{n=0}^{\infty} q^{n/2} R'_n = \bigotimes_{\ell=\frac{1}{2}, \frac{3}{2}, \dots}^{\infty} \lambda_{q^\ell}(T) \bigotimes_{m=1}^{\infty} S_{q^m}(T)$$

Now suppose  $M$  is a  $2n$  dimensional compact Riemannian Spin manifold and denote by  $D^+$  the Dirac operator of  $M$ . For each  $n$  we may form the operators

$$D^+ \otimes (E^+ \oplus E^-) \otimes R_n \quad \text{and} \quad D^+ \otimes R'_n.$$

**THEOREM 6.1.** *These operators are rigid under any  $S^1$  action on  $M$  by isometries, i.e. the induced action on the index of any of these operators is the trivial action.*

This is the theorem conjectured by Witten and first proven by Taubes [T]. A beautiful proof of it appears in [BT].

Roughly speaking Bott and Taubes' proof goes as follows. First they show that the Signature operator  $d_S = D^+ \otimes (E^+ \oplus E^-)$  is rigid by an argument similar to that presented above. Combining this result with the power series  $R(q)$  and interpreting  $\text{ch}(\text{Index}(d_S \otimes R(q)))$  as a meromorphic function on the complex torus  $T_{q^2} = \mathbf{C}^*/q^2$ , they show that it has poles only at roots of unity and no poles on a certain circle  $S^1 \subset T_{q^2}$ . The Spin hypothesis then implies

that it has no poles at all and hence is constant. Thus the character of  $S^1$  given by its action on  $\text{Index}(d_S \otimes R(q))$  is constant, and so the action must be trivial as claimed. They then give separate arguments to extend this result to  $D^+ \otimes R'_n$ .

These results all extend in a straight forward way to  $S^1$  actions preserving a foliation (see [HL 2]).

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