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4. THE LEFSCHETZ THEOREM FOR FOLIATED MANIFOLDS

Let M be a compact m dimensional manifold and F a dimension n foliation on M . Then F is an n dimensional subbundle of TM such that for any two sections $X, Y \in C^\infty(F)$, $[X, Y] \in C^\infty(F)$. The Frobenius Theorem says that for each $x \in M$, there is a neighborhood U of x and a diffeomorphism

$$\phi : \mathbf{R}^n \times \mathbf{R}^q \rightarrow U \quad n + q = m$$

so that for all $z \in \mathbf{R}^n \times \mathbf{R}^q$.

$$d\phi(T\mathbf{R}_z^n) = F_{\phi(z)}.$$

Such a (U, ϕ) is called a foliation chart. Given $x \in \mathbf{R}^q$, the submanifold $\phi(\mathbf{R}^n \times \{x\})$ is called a plaque, and is denoted P_x^U . It is a local integral submanifold of F . The submanifold $\phi(\{0\} \times \mathbf{R}^q)$ is denoted \mathbf{R}_U^q and is called the transverse submanifold of (U, ϕ) .

A leaf L of F is a maximal integral (i.e. $TL_x = F_x$ for all $x \in L$) submanifold of M . Thus $\dim L = n$. The Frobenius Theorem implies that through each point x in M , there passes a unique leaf, denoted L_x . Each leaf is a complete manifold of bounded geometry and the bounds are uniform for all leaves.

We now extend the Lefschetz Theorem for compact manifolds to a Lefschetz Theorem for foliations of a compact manifold. This is joint work with Connor Lazarov [HL 1]. In fact, we show how to improve the results of [HL 1] by removing the assumption that F admits a transverse invariant metric. For a K-theory version of this result, see the thesis of M-T. Benameur, [Be].

Choose a smooth metric on M . This induces a smooth metric on each leaf L , and L is complete with respect to this metric. Two different metrics on M induce quasi-isometric metrics on L .

HAEFLIGER FORMS

Let $\{U_i\}$ be a finite cover of M by foliation charts. For $x \in U_i$, denote its plaque by P_x^i . If $U_i \cap U_j \neq \emptyset$ we define a local diffeomorphism f_{ij} from $\mathbf{R}_{U_i}^q$ (hereafter denoted \mathbf{R}_i^q) to \mathbf{R}_j^q as follows:

$$f_{ij}(x) = y \text{ if and only if } P_x^i \cap P_y^j \neq \emptyset.$$

The f_{ij} generate the holonomy pseudogroup, denoted H , which acts on the transversal space $T = \cup_i \mathbf{R}_i^q$. We may (and do) assume that the \mathbf{R}_i^q are disjoint.

Recall the following construction due to Haefliger [Ha]. Let $\Omega_c^k(T)$ be the space of bounded measurable complex valued k forms on T with compact support. Denote by $\Omega_c^k(T/H)$ the quotient of $\Omega_c^k(T)$ by the vector subspace generated by elements of the form $\alpha - h^*\alpha$ where $h \in H$ and $\alpha \in \Omega_c^k(T)$ has support contained in the range of h . Give $\Omega_c^k(T/H)$ the quotient topology of the usual sup norm topology on $\Omega_c^k(T)$. Note that $\Omega_c^k(T/H)$ does not depend of the choice of cover used to define it.

Denote by $\Omega^{p+k}(M)$ the space of bounded measurable complex valued $p+k$ forms on M . As the bundle TF is oriented, there is a continuous open surjective linear map,

$$\int_F : \Omega^{p+k}(M) \rightarrow \Omega_c^k(T/H).$$

It is given as follows. Let $\omega \in \Omega^{p+k}(M)$ and let $\{\psi_i\}$ be a partition of unity subordinate to the cover $\{U_i\}$. Set $\omega_i = \psi_i\omega$. We may integrate ω_i along the fibers of the submersion $\pi_i : U_i \rightarrow \mathbf{R}_i^q$ to obtain $\bar{\omega}_i \in \Omega_c^k(\mathbf{R}_i^q)$. Define $\int_F \omega$ to be the class of $\sum \bar{\omega}_i$ in $\Omega_c^k(T/H)$. It is independent of the choices made in defining it.

DIFFERENTIAL COMPLEXES ON M ELLIPTIC ALONG F

A differential complex on M along F consists of :

- a) a finite collection of finite dimensional complex vector bundles E_0, \dots, E_k over M
- b) a collection of smooth differential operators

$$d_i : C^\infty(E_i) \rightarrow C^\infty(E_{i+1})$$

with $d_{i+1} \cdot d_i = 0$

- c) each d_i differentiates only in leaf directions.

For the sake of simplicity we assume that each d_i is first order.

Each of the classical complexes mentioned above (de Rham, Dolbeault, Signature and Twisted Spin) gives a leafwise complex on M provided that the leaves have the required structures and that these structures are coherent from leaf to leaf (i.e. come from a global structure on M). For example, in the twisted Spin case, we require that the Spin structure on the leaves comes from a principal Spin(n) bundle P over M with $P \times_{\text{Spin}(n)} \mathbf{R}^n \simeq TF$, and that the leafwise auxiliary twisting bundle come from a bundle over M .

For a fixed leaf L , denote $E_i|_L$ by E_i^L and by $C_0^\infty(E_i^L)$ the space of smooth sections of E_i^L with compact support. The operator d_i induces one, denoted also by d_i ,

$$d_i : C_0^\infty(E_i^L) \rightarrow C_0^\infty(E_{i+1}^L)$$

and on L we have the complex

$$0 \rightarrow C_0^\infty(E_0^L) \xrightarrow{d_0} C_0^\infty(E_1^L) \xrightarrow{d_1} \dots \xrightarrow{d_{k-1}} C_0^\infty(E_k^L) \rightarrow 0.$$

We say that the complex (E, d) is elliptic along F provided that for each leaf L , the above complex is elliptic. We assume that (E, d) is elliptic along F .

L^2 COHOMOLOGY OF (E, d)

Choose a smooth Hermitian metric on each bundle E_i over M . These induce metrics on each E_i^L and these metrics are unique up to quasi-isometry. Using these metrics we construct $d_i^* : C_0^\infty(E_{i+1}^L) \rightarrow C_0^\infty(E_i^L)$ just as we did before. We then construct

$$\Delta_i^L : C_0^\infty(E_i^L) \rightarrow C_0^\infty(E_i^L)$$

and we extend Δ_i to

$$\Delta_i^L : L^2(E_i^L) \rightarrow L^2(E_i^L)$$

just as before.

DEFINITION 4.1. *The i th L^2 cohomology of (E, d) along the leaf L , denoted $H_L^i(E, d)$ is*

$$H_L^i(E, d) = \ker \Delta_i^L.$$

The i th L^2 cohomology of (E, d) is denoted $H^i(E, d)$ and it assigns to each leaf L the i th cohomology of (E, d) along L , $H_L^i(E, d)$.

SOME FACTS

1. $H_L^i(E, d)$ consists of smooth sections and $\dim_{\mathbb{C}} H_L^i(E, d)$ may be infinite but is always countable.
2. π_L^i , the projection of $L^2(E_i^L)$ onto $H_L^i(E, d)$, is a smoothing operator (on L) with smooth Schwartz kernel $k_L^i(x, y)$.
3. $k_L^i(x, y)$ is measurable as a function of L and bounded independently of L . In particular, $\text{tr } k_L^i(x, x)$ is a bounded measurable function on M whose restriction to each leaf L is smooth.
4. Because of 3. above, we may define the dimension of $H^i(E, d)$ to be the zero dimensional Haefliger form

$$\dim(H^i(E, d)) = \int_F \operatorname{tr}(k_L^i(x, x)) dx,$$

where for any leaf L we denote the volume form obtained from the metric on L by dx . We may also define the Euler class of (E, d) as

$$\chi(E, d) = \sum_{i=0}^k (-1)^i \dim H^i(E, d).$$

GEOMETRIC ENDOMORPHISMS

Let $f : M \rightarrow M$ be a diffeomorphism and assume that for each leaf L of F , $f(L) \subset L$. For each i , let

$$A_i : f^* E_i \rightarrow E_i$$

be a smooth bundle map. We assume that $T_i : C^\infty(E_i) \rightarrow C^\infty(E_i)$ where $(T_i s)(x) = A_{i,x} s(f(x))$ satisfies

$$T_i d_{i-1} = d_{i-1} T_{i-1}.$$

The T_i then induce maps

$$T_i^L : C_0^\infty(E_i^L) \rightarrow C_0^\infty(E_i^L)$$

satisfying

$$T_i^L d_{i-1} = d_{i-1} T_{i-1}^L.$$

We call such a family $T = (T_0, \dots, T_k)$ the geometric endomorphism of (E, d) defined by f and $A = (A_0, \dots, A_k)$. The T_i^L extend to uniformly bounded linear maps

$$T_i^L : L^2(E_i^L) \rightarrow L^2(E_i^L).$$

LEFSCHETZ NUMBER OF A GEOMETRIC ENDOMORPHISM

Set $T_{i,L}^* = \pi_i^L \cdot T_i^L \cdot \pi_i^L$ and denote its Schwartz kernel by $k_L^{T_i^*}(x, y)$. Then $k_L^{T_i^*}(x, y)$ is globally bounded, smooth on $L \times L$, and measurable. Thus $\operatorname{tr}(k_L^{T_i^*}(x, x))$ is a bounded measurable function on M which is smooth on each leaf L . We define the Lefschetz class of the geometric endomorphism T to be the Haefliger zero form

$$L(T) = \sum_{i=0}^k (-1)^i \int_F \operatorname{tr}(k_L^{T_i^*}(x, x)) dx.$$

For our Lefschetz Theorem we shall also need two restrictions on the fixed point set, N of f . We require :

1. $N = \bigcup_{\alpha} N_{\alpha}$ is a finite disjoint union of closed, connected submanifolds N_{α} , each transverse to F .
2. for each $x \in N \cap L = \bigcup_{\alpha} N_{\alpha}^L$ where $N_{\alpha}^L = N_{\alpha} \cap L$, df_x has no eigen vector (in TL_x) with eigenvalue $+1$ in directions transverse (in $L!$) to N_{α}^L .

Note in particular that $f = id_M$ satisfies these conditions.

FIXED POINT INDICES

Let $\{U_i\}$ and $\{\psi_i\}$ be as above. Suppose that for each L and α we are given a differential form a_{α}^L defined on N_{α}^L . We define the Haefliger form $\int_N a$ as

$$\int_N a = \sum_i \sum_{N_{\alpha}^L \cap P_x^i \neq \phi} \int_{N_{\alpha}^L \cap P_x^i} \psi_i a_{\alpha}^L.$$

Note that for any plaque P_x^i , only a finite number of N_{α}^L satisfy $N_{\alpha}^L \cap P_x^i \neq \phi$. As $\int_{N_{\alpha}^L \cap P_x^i} \psi_i a_{\alpha}^L$ is a differential form on the transversal \mathbf{R}_i^q of U_i , we may also consider it as a Haefliger form for F . As above, it is not difficult to show that the Haefliger form $\int_N a$ does not depend on the choices made in defining it.

THEOREM 4.2 (The Lefschetz Theorem for Foliations [HL 1]). *Let M , F , f , T , A and (E, d) be as above. To each $N_{\alpha}^L \subset N$ we may associate a differential form a_{α}^L which depends only on local data on N_{α}^L so that*

$$L(T) = \int_N a.$$

The proof follows the outline given above for the classical case, done leafwise. There are some very formidable technical obstacles, but these can be overcome (see [HL 1]).

If (E, d) is the de Rham, Dolbeault, Signature or Twisted Spin complex of F , and $f = id_M$, and $T = id$, then a_j^L is the usual local integrand formula (computed on each leaf, not on M) given by the Atiyah-Singer Index Theorem. We thus have an index theorem for foliated manifolds for these operators.

(Note that Connes has also proven an index theorem for foliated manifolds, (see [C]). As he works on the holonomy coverings of the leaves of F , his theorem is related to ours as the L^2 covering index theorem is related to the ordinary index theorem.) If we take the codimension 0 foliation of M which has one leaf (namely M), we recover the Atiyah-Singer Index Theorem for these operators. In general, i.e. $f \neq id_M$, $T = f^*$, a_j^L is the usual local integrand (computed on the fixed point set in each leaf, not in M) given by the Atiyah-Singer G Index Theorem. If we take the codimension 0 foliation, we recover the Atiyah-Singer G Index Theorem and the Atiyah-Bott Lefschetz Theorem for these operators.

5. GROUP ACTIONS AND THE LEFSCHETZ THEOREM

Let F be an oriented $2k$ dimensional foliation of a compact, oriented, Riemannian manifold M . Assume that F admits a $\text{Spin}(2k)$ structure. That is, there is a principal $\text{Spin}(2k)$ bundle P over M and an isomorphism of oriented bundles

$$P \times_{\text{Spin}(2k)} \mathbf{R}^{2k} \simeq TF.$$

We may then construct the bundles $E^\pm = P \times_{\text{Spin}(2k)} \Delta^\pm$. The leafwise Dirac operator D^+ is constructed using the Riemannian structure on the leaves of F which is induced from M .

Let G be a compact, connected Lie group acting by isometries on M , taking each leaf of F to itself. G then acts on TF . We assume that G also acts on P (commuting with the action of $\text{Spin}(2k)$) so that the induced action on $P \times_{\text{Spin}(2k)} \mathbf{R}^{2k} \simeq TF$ is the given action on TF . G then acts on the bundles E^\pm and it commutes with the operator D^+ , i.e. G is a group of geometric endomorphisms of the complex (E^\pm, D^+) .

Recall the \widehat{A} genus defined in Section 1.

DEFINITION 5.1. *The \widehat{A} genus of F is the Haefliger zero form*

$$\widehat{A}(F) = \int_F \widehat{A}_{k/2}(TF).$$

In particular, if k is odd, $\widehat{A}(F) = 0$.

Note that we have defined $\widehat{A}(F)$ as the zero th order part of $\int_F \widehat{A}(TF)$. For an interpretation of the higher order terms of $\int_F \widehat{A}(TF)$, see [He].