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2. THE INDEX OF AN ELLIPTIC COMPLEX

An *elliptic complex* (E, d) over a closed, oriented, n dimensional Riemannian manifold M consists of:

- a) a finite collection of finite dimensional complex vector bundles

$$E_0, E_1, \dots, E_k$$

- b) a collection of smooth differential operators

$$d_i : C^\infty(E_i) \rightarrow C^\infty(E_{i+1})$$

- c) The operators d_i are required to satisfy

$$d_{i+1} \cdot d_i = 0$$

and an additional technical condition called ellipticity. This condition makes possible the construction a virtual bundle, i.e. the formal difference of two vector bundles, over TM which carries a great deal of information about (E, d) . This virtual bundle $\sigma(E, d)$ is called the symbol of (E, d) and it defines a class $[\sigma(E, d)]$, also called the symbol, in the K theory with compact supports of TM .

EXAMPLES

1. The de Rham complex, where

$T_{\mathbb{C}}^*M$ = complexified cotangent bundle of M

$E_i = \Lambda^i T_{\mathbb{C}}^*M$ the i th exterior power of $T_{\mathbb{C}}^*M$

$C^\infty(E_i)$ = smooth complex i forms on M

d_i = the usual exterior derivative

2. The Dolbeault complex
 3. The Signature complex (see [AS])
 4. The twisted Spin complex.

SOME FACTS ABOUT ELLIPTIC COMPLEXES

Set $H^i(E, d) = \ker d_i / \text{image } d_{i-1}$. If M is compact, then $\dim H^i(E, d) < \infty$, and we may define

$$\text{Index}(E, d) = \sum_{i=0}^k (-1)^i \dim H^i(E, d).$$

This is a very important invariant. Special cases of (E, d) yield the

1. Euler class $\chi(M)$ of M (de Rham complex)
2. Signature of M (Signature complex)
3. Euler class $\chi(M, V)$ (Dolbeault complex)
4. \hat{A} genus of M (Spin complex).

The Atiyah-Singer Index Theorem tells how to compute this invariant from topological information about M and (E, d) . In particular, it says

THEOREM 2.1 ([AS]).

$$\text{Index}(E, d) = \int_M Td(TM \otimes_{\mathbf{R}} \mathbf{C}) \cdot \text{ch}(\sigma(E, d)).$$

The theorems quoted above are all special cases of this theorem. We now give an idea of how to prove this deep and important theorem.

On each E_i choose an Hermitian inner product denoted $(\ , \)_i$. This induces an inner product $\langle \ , \ \rangle_i$ on $C^\infty(E_i)$ by the formula

$$\langle s_1, s_2 \rangle_i = \int_M (s_1(x), s_2(x))_i dx.$$

Using $\langle \ , \ \rangle_i$ we define the adjoints

$$d_i^* : C^\infty(E_i) \rightarrow C^\infty(E_{i-1})$$

by

$$\langle s_1, d_i^* s_2 \rangle_{i-1} = \langle d_{i-1} s_1, s_2 \rangle_i$$

where

$$s_1 \in C^\infty(E_{i-1}), \quad s_2 \in C^\infty(E_i).$$

The Laplacian $\Delta_i : C^\infty(E_i) \rightarrow C^\infty(E_i)$ is defined by

$$\Delta_i = d_{i-1} d_i^* + d_{i+1}^* d_i,$$

and it extends to a densely defined operator of $L^2(E_i)$, the space of L^2 sections of E_i , as follows. Δ_i is a diagonalizable operator, and any eigenvalue λ of Δ_i must be real and nonnegative. If M is compact there is a sequence of real numbers

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \quad \lim_{j \rightarrow \infty} \lambda_j = \infty$$

such that for each E_i there is a sequence of finite dimensional subspaces of $C^\infty(E_i)$, denoted

$$E_i(\lambda_0), E_i(\lambda_1), E_i(\lambda_2), \dots$$

so that for any $s \in E_i(\lambda_j)$

$$\Delta_i s = \lambda_j s.$$

In addition

$$L^2(E_i) = \bigoplus_{j=0}^{\infty} E_i(\lambda_j).$$

Thus each element in $L^2(E_i)$ can be written as a (possibly infinite) sum of eigen functions and we may think of Δ_i as the infinite diagonal matrix

$$\begin{bmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & \lambda_1 & & & \\ & & & & \ddots & & \\ & & & & & \lambda_1 & \\ & & & & & & \lambda_2 \\ & & & & & & & \ddots \\ & & & & & & & & \lambda_2 \\ & & & & & & & & & \ddots \end{bmatrix}.$$

OTHER PROPERTIES OF Δ_i

- 1) $E_i(\lambda_0) = \ker \Delta_i \subset \ker d_i$ and the inclusion of $E_i(\lambda_0)$ in $\ker d_i$ induces an isomorphism

$$E_i(\lambda_0) \simeq H^i(E, d),$$

so

$$\text{Index}(E, d) = \sum_{i=0}^k (-1)^i \dim E_i(\lambda_0).$$

2) For each $\lambda_j > 0$, the sequence

$$0 \rightarrow E_0(\lambda_j) \xrightarrow{d_0} E_1(\lambda_j) \xrightarrow{d_1} \cdots \xrightarrow{d_{k-1}} E_k(\lambda_j) \rightarrow 0$$

is exact.

As a corollary, we have immediately

$$\sum_{i=0}^k (-1)^i \dim E_i(\lambda_j) = 0$$

for all $\lambda_j > 0$. These results rely on the fact that M is compact. For a general reference for the above facts, see [Wa].

The fact that Δ_i is diagonal implies that for any function $f : \mathbf{R} \rightarrow \mathbf{R}$, we may define

$$f(\Delta_i) : L^2(E_i) \rightarrow L^2(E_i)$$

by : for each $s \in E_i(\lambda_j)$ set $f(\Delta_i)s = f(\lambda_j)s$. i.e. the “matrix” of $f(\Delta_i)$ is

$$\begin{bmatrix} f(0) & & & & & \\ & \ddots & & & & \\ & & f(0) & & & \\ & & & f(\lambda_1) & & \\ & & & & \ddots & \\ & & & & & f(\lambda_1) \\ & & & & & & \ddots \end{bmatrix}$$

Note also that if $f(x)$ goes to zero rapidly enough as $x \rightarrow \infty$, then the trace of $f(\Delta_i)$, thought of as the usual trace applied to the infinite matrix above, will be a finite number. In this case, we say $f(\Delta_i)$ is of trace class. See [RS].

We are interested in the family of functions $f_t(x) = e^{-tx}$, $t > 0$. In fact, even if M is not compact, $e^{-t\Delta}$ makes sense and we have

THEOREM 2.2 (Seeley, [S]). *For $t > 0$, $e^{-t\Delta_i}$ is a smoothing operator on $L^2(E_i)$ and so if M is compact it is of trace class.*

Let $\pi_j : M \times M \rightarrow M$ be projection on the j th factor, $j = 1, 2$. To say an operator A on $L^2(E_i)$ is a smoothing operator means that there is a smooth section $k(x, y)$ of the bundle $\text{Hom}(\pi_2^* E_i, \pi_1^* E_i)$ over $M \times M$, so that for all $s \in L^2(E_i)$.

$$(As)(x) = \int_M k(x, y)s(y)dy.$$

Note that $k(x, y)$ is a linear map from $E_{i,y}$, the fiber over y , to $E_{i,x}$, the fiber over x , so $k(x, x) : E_{i,x} \rightarrow E_{i,x}$ has a well defined trace. The section $k(x, y)$ is called the Schwartz kernel of A . Any smoothing operator on a compact manifold is of trace class and its trace is given by $\text{tr}(A) = \int_M \text{tr } k(x, x)dx$.

To see this for $e^{-t\Delta_i}$, note that its Schwartz kernel $k_t^i(x, y)$ must be given as follows: For each λ_j choose an orthonormal basis ϕ_j^v , $v = 1, \dots, \dim E_i(\lambda_j)$ of $E_i(\lambda_j)$. Then

$$k_t^i(x, y) = \sum_{j=0}^{\infty} e^{-t\lambda_j} \left[\sum_v \phi_j^v(x) \phi_j^v(y) \right].$$

Here $k_t^i(x, y) : E_{i,y} \rightarrow E_{i,x}$ acts on $w \in E_{i,y}$ by

$$k_t^i(x, y)w = \sum_{j=0}^{\infty} e^{-t\lambda_j} \left[\sum_v (\phi_j^v(y), w)_i \cdot \phi_j^v(x) \right].$$

The trace of $k_t^i(x, x)$ is then given by

$$\sum_{j=0}^{\infty} e^{-t\lambda_j} \left[\sum_v (\phi_j^v(x), \phi_j^v(x))_i \right]$$

and the result follows by integrating over M .

Now, since $e^{-t\lambda_0} = 1$ for all t , we have $e^{-t\lambda_0} \sum_{i=0}^k (-1)^i \dim E_i(\lambda_0) =$

$\text{Index}(E, d)$, for all t . In addition $e^{-t\lambda_j} \sum_{i=0}^k (-1)^i \dim E_i(\lambda_j) = 0$ for $j > 0$, and for all t . Thus we have

THEOREM 2.3. *If M is compact, then for all $t > 0$,*

$$\begin{aligned} \text{Index}(E, d) &= \sum_{j=0}^{\infty} \left[\sum_{i=0}^k (-1)^i e^{-t\lambda_j} \dim E_i(\lambda_j) \right] \\ &= \sum_{i=0}^k \left[\sum_{j=0}^{\infty} (-1)^i e^{-t\lambda_j} \dim E_i(\lambda_j) \right] \\ &= \sum_{i=0}^k (-1)^i \text{tr } e^{-t\Delta_i}. \end{aligned}$$

The Index Theorem now follows from two other results.

- 1) Set $k_t(x) = \sum_{i=0}^k (-1)^i \text{tr } k_t^i(x, x)$. Then for t near 0, $k_t(x)$ has an asymptotic expansion of the form

$$k_t(x) = \sum_{j \geq -n} t^{j/2} a_j(x).$$

As $\int_M k_t(x) dx = \sum_{i=0}^k (-1)^i \text{tr } e^{-t\Delta_i} = \text{Index}(E, d)$ is independent of t , we have

$$\text{Index}(E, d) = \int_M a_0(x) dx.$$

Now, for any twisted Dirac operator D_F^+ , one can prove that the differential n form $a_0(x)dx$ is the degree n part of the form constructed from the connections on TM and F which represents $\hat{A}(TM) \cdot ch(F)$. Thus, we have

$$\text{Index}(D_F^+) = \int_M \hat{A}(TM) \cdot ch(F).$$

- 2) $\text{Index}(E, d)$ depends only on the K theory class $[\sigma(E, d)]$. Given this and the formula above for $\text{Index}(D_F^+)$, one may use well known arguments in K theory to extend the result in 1. to all elliptic complexes. The essential fact is that the symbols of twisted Dirac operators generate the K theory with compact supports of TM as an algebra over the K theory of M .

The difference between the formula in 1. and that in the Atiyah-Singer Index Theorem is accounted for by the fact that for the twisted Spin complex $(E^\pm \otimes F, D_F^+)$,

$$ch(\sigma(E^\pm \otimes F, D_F^+)) = ch(E^\pm, D^+) \cdot ch(F)$$

and

$$Td(TM \otimes_{\mathbf{R}} \mathbf{C}) \cdot \mathbf{ch}(\sigma(E^\pm, D^+)) = \hat{A}(TM).$$

For more on this see [ABP], [B], [G], [Gi], and [P].