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Autor:	HEITSCH, James L.
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2. The index of an elliptic complex

An *elliptic complex* (E, d) over a closed, oriented, *n* dimensional Riemannian manifold *M* consists of :

a) a finite collection of finite dimensional complex vector bundles

$$E_0, E_1 \ldots, E_k$$

b) a collection of smooth differential operators

$$d_i: C^{\infty}(E_i) \to C^{\infty}(E_{i+1})$$

c) The operators d_i are required to satisfy

$$d_{i+1} \cdot d_i = 0$$

and an additional technical condition called ellipticity. This condition makes possible the construction a virtual bundle, i.e. the formal difference of two vector bundles, over *TM* which carries a great deal of information about (E, d). This virtual bundle $\sigma(E, d)$ is called the symbol of (E, d) and it defines a class $[\sigma(E, d)]$, also called the symbol, in the K theory with compact supports of *TM*.

EXAMPLES

1. The de Rham complex, where

 $T^*_{\mathbf{C}}M = \text{complexified cotangent bundle of } M$ $E_i = \Lambda^i T^*_{\mathbf{C}}M$ the *ith* exterior power of $T^*_{\mathbf{C}}M$ $C^{\infty}(E_i) = \text{smooth complex } i \text{ forms on } M$ $d_i = \text{the usual exterior derivative}$

- 2. The Dolbeault complex
- 3. The Signature complex (see [AS])
- 4. The twisted Spin complex.

SOME FACTS ABOUT ELLIPTIC COMPLEXES

Set $H^i(E,d) = \ker d_i/\operatorname{image} d_{i-1}$. If *M* is compact, then dim $H^i(E,d) < \infty$, and we may define

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$$(E, d) = \sum_{i=0}^{k} (-1)^{i} \dim H^{i}(E, d)$$
.

This is a very important invariant. Special cases of (E, d) yield the

- 1. Euler class $\chi(M)$ of M (de Rham complex)
- 2. Signature of M (Signature complex)
- 3. Euler class $\chi(M, V)$ (Dolbeault complex)
- 4. $\widehat{\mathcal{A}}$ genus of *M* (Spin complex).

The Atiyah-Singer Index Theorem tells how to compute this invariant from topological information about M and (E, d). In particular, it says

THEOREM 2.1 ([AS]).

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$$(E, d) = \int_{M} Td(TM \otimes_{\mathbf{R}} \mathbf{C}) \cdot \operatorname{ch} \left(\sigma(E, d)\right).$$

The theorems quoted above are all special cases of this theorem. We now give an idea of how to prove this deep and important theorem.

On each E_i choose an Hermitian inner product denoted $(,)_i$. This induces an inner product \langle , \rangle_i on $C^{\infty}(E_i)$ by the formula

$$\langle s_1, s_2 \rangle_i = \int_M \left(s_1(x), s_2(x) \right)_i dx.$$

Using \langle , \rangle_i we define the adjoints

$$d_i^*: C^{\infty}(E_i) \to C^{\infty}(E_{i-1})$$

by

$$\langle s_1, d_i^* s_2 \rangle_{i-1} = \langle d_{i-1} s_1, s_2 \rangle_i$$

where

$$s_1 \in C^{\infty}(E_{i-1}), \quad s_2 \in C^{\infty}(E_i).$$

The Laplacian $\Delta_i : C^{\infty}(E_i) \to C^{\infty}(E_i)$ is defined by

 $\Delta_i = d_{i-1}d_i^* + d_{i+1}^*d_i \,,$

and it extends to a densely defined operator of $L^2(E_i)$, the space of L^2 sections of E_i , as follows. Δ_i is a diagonalizable operator, and any eigenvalue λ of Δ_i must be real and nonnegative. If M is compact there is a sequence of real numbers

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \quad \lim_{j \to \infty} \lambda_j = \infty$$

such that for each E_i there is a sequence of finite dimensional subspaces of $C^{\infty}(E_i)$, denoted

$$E_i(\lambda_0), E_i(\lambda_1), E_i(\lambda_2), \ldots$$

so that for any $s \in E_i(\lambda_j)$

$$\Delta_i s = \lambda_j s$$

In addition

$$L^2(E_i) = \bigoplus_{j=0}^{\infty} E_i(\lambda_j).$$

Thus each element in $L^2(E_i)$ can be written as a (possibly infinite) sum of eigen functions and we may think of Δ_i as the infinite diagonal matrix



1) $E_i(\lambda_0) = \ker \Delta_i \subset \ker d_i$ and the inclusion of $E_i(\lambda_0)$ in $\ker d_i$ induces an isomorphism

$$E_i(\lambda_0) \simeq H^i(E, d)$$
,

SO

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$$(E, d) = \sum_{i=0}^{k} (-1)^{i} \dim E_{i}(\lambda_{0})$$



2) For each $\lambda_i > 0$, the sequence

$$0 \to E_0(\lambda_j) \xrightarrow{d_0} E_1(\lambda_j) \xrightarrow{d_1} \cdots \xrightarrow{d_{k-1}} E_k(\lambda_j) \to 0$$

is exact.

As a corollary, we have immediately

$$\sum_{i=0}^{k} (-1)^{i} \dim E_{i}(\lambda_{j}) = 0$$

for all $\lambda_j > 0$. These results rely on the fact that *M* is compact. For a general reference for the above facts, see [Wa].

The fact that Δ_i is diagonal implies that for any function $f : \mathbf{R} \to \mathbf{R}$, we may define

$$f(\Delta_i): L^2(E_i) \to L^2(E_i)$$

by: for each $s \in E_i(\lambda_j)$ set $f(\Delta_i)s = f(\lambda_j)s$. i.e. the "matrix" of $f(\Delta_i)$ is



Note also that if f(x) goes to zero rapidly enough as $x \to \infty$, then the trace of $f(\Delta_i)$, thought of as the usual trace applied to the infinite matrix above, will be a finite number. In this case, we say $f(\Delta_i)$ is of trace class. See [RS].

We are interested in the family of functions $f_t(x) = e^{-tx}$, t > 0. In fact, even if M is not compact, $e^{-t\Delta}$ makes sense and we have

THEOREM 2.2 (Seeley, [S]). For t > 0, $e^{-t\Delta_i}$ is a smoothing operator on $L^2(E_i)$ and so if M is compact it is of trace class.

Let $\pi_j : M \times M \to M$ be projection on the *j*th factor, j = 1, 2. To say an operator A on $L^2(E_i)$ is a smoothing operator means that there is a smooth section k(x, y) of the bundle $\text{Hom}(\pi_2^* E_i, \pi_1^* E_i)$ over $M \times M$, so that for all $s \in L^2(E_i)$.

$$(As)(x) = \int_{M} k(x, y)s(y)dy.$$

Note that k(x, y) is a linear map from $E_{i,y}$, the fiber over y, to $E_{i,x}$, the fiber over x, so $k(x,x) : E_{i,x} \to E_{i,x}$ has a well defined trace. The section k(x,y)is called the Schwartz kernel of A. Any smoothing operator on a compact manifold is of trace class and its trace is given by $tr(A) = \int_{M} tr \ k(x,x) dx$. To see this for $e^{-t\Delta_i}$, note that its Schwartz kernel $k_i^i(x,y)$ must be given as follows: For each λ_j choose on orthonormal basis ϕ_j^v , $v = 1, \ldots$, dim $E_i(\lambda_j)$ of $E_i(\lambda_j)$. Then

$$k_t^i(x,y) = \sum_{j=0}^{\infty} e^{-t\lambda_j} \left[\sum_{v} \phi_j^v(x) \phi_j^v(y) \right] \,.$$

Here $k_t^i(x, y) : E_{i,y} \to E_{i,x}$ acts on $w \in E_{i,y}$ by

$$k_t^i(x, y)w = \sum_{j=0}^{\infty} e^{-t\lambda_j} \left[\sum_{v} \left(\phi_j^v(y), w \right)_i \cdot \phi_j^v(x) \right]$$

The trace of $k_t^i(x, x)$ is then given by

$$\sum_{j=0}^{\infty} e^{-t\lambda_j} \left[\sum_{v} \left(\phi_j^v(x), \phi_j^v(x) \right)_i \right]$$

and the result follows by integrating over M.

Now, since $e^{-t\lambda_0} = 1$ for all t, we have $e^{-t\lambda_0} \sum_{i=0}^k (-1)^i \dim E_i(\lambda_0) =$ Index (E, d), for all t. In addition $e^{-t\lambda_j} \sum_{i=0}^k (-1)^i \dim E_i(\lambda_j) = 0$ for j > 0, and for all t. Thus we have

THEOREM 2.3. If M is compact, then for all t > 0,

$$Index(E, d) = \sum_{j=0}^{\infty} \left[\sum_{i=0}^{k} (-1)^{i} e^{-t\lambda_{j}} \dim E_{i}(\lambda_{j}) \right]$$
$$= \sum_{i=0}^{k} \left[\sum_{j=0}^{\infty} (-1)^{i} e^{-t\lambda_{j}} \dim E_{i}(\lambda_{j}) \right]$$
$$= \sum_{i=0}^{k} (-1)^{i} tr \ e^{-t\Delta_{i}}.$$

The Index Theorem now follows from two other results.

1) Set $k_t(x) = \sum_{i=0}^{k} (-1)^i \text{tr } k_t^i(x, x)$. Then for t near 0, $k_t(x)$ has an asymptotic expansion of the form

$$k_t(x) = \sum_{j \ge -n} t^{j/2} a_j(x) \, .$$

As $\int_{M} k_t(x) dx = \sum_{i=0}^{k} (-1)^i \text{tr } e^{-t\Delta_i} = \text{Index}(E, d)$ is independent of t, we have

Index
$$(E, d) = \int_{M} a_0(x) dx$$
.

Now, for any twisted Dirac operator D_F^+ , one can prove that the differential n form $a_0(x)dx$ is the degree n part of the form constructed from the connections on TM and F which represents $\widehat{\mathcal{A}}(TM) \cdot ch(F)$. Thus, we have

Index
$$(D_F^+) = \int_M \widehat{\mathcal{A}}(TM) \cdot ch(F)$$
.

2) Index (E, d) depends only on the K theory class $[\sigma(E, d)]$. Given this and the formula above for Index (D_F^+) , one may use well known arguments in K theory to extend the result in 1. to all elliptic complexes. The essential fact is that the symbols of twisted Dirac operators generate the K theory with compact supports of TM as an algebra over the K theory of M.

The difference between the formula in 1. and that in the Atiyah-Singer Index Theorem is accounted for by the fact that for the twisted Spin complex $(E^{\pm} \otimes F, D_F^{+})$,

$$ch(\sigma(E^{\pm}\otimes F, D_F^{+})) = ch(E^{\pm}, D^{+}) \cdot ch(F)$$

and

$$Td(TM \otimes_{\mathbf{R}} \mathbf{C}) \cdot \mathbf{ch} (\sigma(E^{\pm}, D^{+})) = \widehat{\mathcal{A}}(TM).$$

For more on this see [ABP], [B], [G], [Gi], and [P].

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