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2. The index of an elliptic complex

An *elliptic complex* (E, d) over a closed, oriented, n dimensional Riemannian manifold M consists of:

a) a finite collection of finite dimensional complex vector bundles

$$E_0, E_1 \ldots, E_k$$

b) a collection of smooth differential operators

$$d_i : C^{\infty}(E_i) \to C^{\infty}(E_{i+1})$$

c) The operators d_i are required to satisfy

$$d_{i+1} \cdot d_i = 0$$

and an additional technical condition called ellipticity. This condition makes possible the construction a virtual bundle, i.e. the formal difference of two vector bundles, over TM which carries a great deal of information about (E, d). This virtual bundle $\sigma(E, d)$ is called the symbol of (E, d) and it defines a class $\left[\sigma(E, d)\right]$, also called the symbol, in the K theory with compact supports of TM.

Examples

1. The de Rham complex, where

$$\begin{aligned} T^*_{\mathbf{C}}M &= \text{complexified cotangent bundle of } M \\ E_i &= \Lambda^i T^*_{\mathbf{C}}M \text{ the } ith \text{ exterior power of } T^*_{\mathbf{C}}M \\ C^{\infty}(E_i) &= \text{smooth complex } i \text{ forms on } M \\ d_i &= \text{the usual exterior derivative} \end{aligned}$$

2. The Dolbeault complex
3. The Signature complex (see [AS])
4. The twisted Spin complex.

SOME FACTS ABOUT ELLIPTIC COMPLEXES

Set $H^i(E,d) = \ker d_i/\text{image } d_{i-1}$. If M is compact, then $\dim H^i(E,d) < \infty$, and we may define

$$\text{Index}(E,d) = \sum_{i=0}^{k} (-1)^i \dim H^i(E,d).$$

This is a very important invariant. Special cases of (E,d) yield the

1. Euler class $\chi(M)$ of M (de Rham complex)
2. Signature of M (Signature complex)
3. Euler class $\chi(M,V)$ (Dolbeault complex)
4. $\widehat{\mathcal{A}}$ genus of M (Spin complex).

The Atiyah-Singer Index Theorem tells how to compute this invariant from topological information about M and (E,d). In particular, it says

THEOREM 2.1 ([AS]).

$$\text{Index}(E,d) = \int_M Td(TM \otimes_{\mathbf{R}} \mathbf{C}) \cdot \text{ch}\left(\sigma(E,d)\right).$$

The theorems quoted above are all special cases of this theorem. We now give an idea of how to prove this deep and important theorem.

On each E_i choose an Hermitian inner product denoted $(\ ,\ )_i$. This induces an inner product $\langle\ ,\ \rangle_i$ on $C^\infty(E_i)$ by the formula

$$\langle s_1, s_2 \rangle_i = \int_M \left(s_1(x), s_2(x)\right)_i dx.$$

Using $\langle\ ,\ \rangle_i$ we define the adjoints

$$d_i^* : C^\infty(E_i) \to C^\infty(E_{i-1})$$

by

$$\langle s_1, d_i^* s_2 \rangle_{i-1} = \langle d_{i-1} s_1, s_2 \rangle_i$$

where

$$s_1 \in C^\infty(E_{i-1}), \quad s_2 \in C^\infty(E_i).$$

The Laplacian $\Delta_i : C^\infty(E_i) \to C^\infty(E_i)$ is defined by

$$\Delta_i = d_{i-1} d_i^* + d_{i+1}^* d_i,$$

and it extends to a densely defined operator of $L^2(E_i)$, the space of L^2 sections of E_i, as follows. Δ_i is a diagonalizable operator, and any eigenvalue λ of Δ_i must be real and nonnegative. If M is compact there is a sequence of real numbers

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots, \quad \lim_{j\to\infty} \lambda_j = \infty$$

such that for each E_i there is a sequence of finite dimensional subspaces of $C^\infty(E_i)$, denoted

$$E_i(\lambda_0), E_i(\lambda_1), E_i(\lambda_2), \ldots$$

so that for any $s \in E_i(\lambda_j)$

$$\Delta_i s = \lambda_j s .$$

In addition

$$L^2(E_i) = \bigoplus_{j=0}^{\infty} E_i(\lambda_j) .$$

Thus each element in $L^2(E_i)$ can be written as a (possibly infinite) sum of eigen functions and we may think of Δ_i as the infinite diagonal matrix

$$\begin{bmatrix} 0 &&&&&&&&& \\ & \ddots &&&&&&&& \\ && 0 &&&&&&& \\ &&& \lambda_1 &&&&&& \\ &&&& \ddots &&&&& \\ &&&&& \lambda_1 &&&& \\ &&&&&& \lambda_2 &&& \\ &&&&&&& \ddots && \\ &&&&&&&& \lambda_2 & \\ &&&&&&&&& \ddots \end{bmatrix} .$$

OTHER PROPERTIES OF Δ_i

1) $E_i(\lambda_0) = \ker \Delta_i \subset \ker d_i$ and the inclusion of $E_i(\lambda_0)$ in $\ker d_i$ induces an isomorphism

$$E_i(\lambda_0) \simeq H^i(E, d),$$

so

$$\mathrm{Index}(E, d) = \sum_{i=0}^{k} (-1)^i \dim E_i(\lambda_0) .$$

2) For each $\lambda_j > 0$, the sequence

$$0 \to E_0(\lambda_j) \xrightarrow{d_0} E_1(\lambda_j) \xrightarrow{d_1} \cdots \xrightarrow{d_{k-1}} E_k(\lambda_j) \to 0$$

is exact.

As a corollary, we have immediately

$$\sum_{i=0}^{k}(-1)^i \dim E_i(\lambda_j) = 0$$

for all $\lambda_j > 0$. These results rely on the fact that M is compact. For a general reference for the above facts, see [Wa].

The fact that Δ_i is diagonal implies that for any function $f : \mathbf{R} \to \mathbf{R}$, we may define

$$f(\Delta_i) : L^2(E_i) \to L^2(E_i)$$

by : for each $s \in E_i(\lambda_j)$ set $f(\Delta_i)s = f(\lambda_j)s$. i.e. the "matrix" of $f(\Delta_i)$ is

$$\begin{bmatrix} f(0) & & & & & & \\ & \ddots & & & & & \\ & & f(0) & & & & \\ & & & f(\lambda_1) & & & \\ & & & & \ddots & & \\ & & & & & f(\lambda_1) & \\ & & & & & & \ddots \end{bmatrix}$$

Note also that if $f(x)$ goes to zero rapidly enough as $x \to \infty$, then the trace of $f(\Delta_i)$, thought of as the usual trace applied to the infinite matrix above, will be a finite number. In this case, we say $f(\Delta_i)$ is of trace class. See [RS].

We are interested in the family of functions $f_t(x) = e^{-tx}$, $t > 0$. In fact, even if M is not compact, $e^{-t\Delta}$ makes sense and we have

THEOREM 2.2 (Seeley, [S]). *For $t > 0$, $e^{-t\Delta_i}$ is a smoothing operator on $L^2(E_i)$ and so if M is compact it is of trace class.*

Let $\pi_j : M \times M \to M$ be projection on the jth factor, $j = 1,2$. To say an operator A on $L^2(E_i)$ is a smoothing operator means that there is a smooth section $k(x,y)$ of the bundle $\mathrm{Hom}(\pi_2^* E_i, \pi_1^* E_i)$ over $M \times M$, so that for all $s \in L^2(E_i)$.

$$(As)(x) = \int_M k(x,y)s(y)dy.$$

Note that $k(x,y)$ is a linear map from $E_{i,y}$, the fiber over y, to $E_{i,x}$, the fiber over x, so $k(x,x) : E_{i,x} \to E_{i,x}$ has a well defined trace. The section $k(x,y)$ is called the Schwartz kernel of A. Any smoothing operator on a compact manifold is of trace class and its trace is given by $\operatorname{tr}(A) = \int_M \operatorname{tr} k(x,x)dx$.

To see this for $e^{-t\Delta_i}$, note that its Schwartz kernel $k_t^i(x,y)$ must be given as follows: For each λ_j choose on orthonormal basis ϕ_j^v, $v = 1, \ldots, \dim E_i(\lambda_j)$ of $E_i(\lambda_j)$. Then

$$k_t^i(x,y) = \sum_{j=0}^{\infty} e^{-t\lambda_j}\left[\sum_v \phi_j^v(x)\phi_j^v(y)\right].$$

Here $k_t^i(x,y) : E_{i,y} \to E_{i,x}$ acts on $w \in E_{i,y}$ by

$$k_t^i(x,y)w = \sum_{j=0}^{\infty} e^{-t\lambda_j}\left[\sum_v \left(\phi_j^v(y), w\right)_i \cdot \phi_j^v(x)\right].$$

The trace of $k_t^i(x,x)$ is then given by

$$\sum_{j=0}^{\infty} e^{-t\lambda_j}\left[\sum_v \left(\phi_j^v(x), \phi_j^v(x)\right)_i\right]$$

and the result follows by integrating over M.

Now, since $e^{-t\lambda_0} = 1$ for all t, we have $e^{-t\lambda_0}\sum_{i=0}^{k}(-1)^i \dim E_i(\lambda_0) =$ Index(E,d), for all t. In addition $e^{-t\lambda_j}\sum_{i=0}^{k}(-1)^i \dim E_i(\lambda_j) = 0$ for $j > 0$, and for all t. Thus we have

THEOREM 2.3. *If M is compact, then for all $t > 0$,*

$$\begin{aligned}\operatorname{Index}(E,d) &= \sum_{j=0}^{\infty}\left[\sum_{i=0}^{k}(-1)^i e^{-t\lambda_j}\dim E_i(\lambda_j)\right]\\ &= \sum_{i=0}^{k}\left[\sum_{j=0}^{\infty}(-1)^i e^{-t\lambda_j}\dim E_i(\lambda_j)\right]\\ &= \sum_{i=0}^{k}(-1)^i \operatorname{tr} e^{-t\Delta_i}.\end{aligned}$$

The Index Theorem now follows from two other results.

1) Set $k_t(x) = \sum_{i=0}^{k}(-1)^i \text{tr } k_t^i(x,x)$. Then for t near 0, $k_t(x)$ has an asymptotic expansion of the form

$$k_t(x) = \sum_{j \geq -n} t^{j/2} a_j(x).$$

As $\int_M k_t(x)dx = \sum_{i=0}^{k}(-1)^i \text{tr } e^{-t\Delta_i} = \text{Index}(E,d)$ is independent of t, we have

$$\text{Index}(E,d) = \int_M a_0(x)dx.$$

Now, for any twisted Dirac operator D_F^+, one can prove that the differential n form $a_0(x)dx$ is the degree n part of the form constructed from the connections on TM and F which represents $\widehat{\mathcal{A}}(TM)\cdot ch(F)$. Thus, we have

$$\text{Index}(D_F^+) = \int_M \widehat{\mathcal{A}}(TM)\cdot ch(F).$$

2) Index(E,d) depends only on the K theory class $[\sigma(E,d)]$. Given this and the formula above for Index(D_F^+), one may use well known arguments in K theory to extend the result in 1. to all elliptic complexes. The essential fact is that the symbols of twisted Dirac operators generate the K theory with compact supports of TM as an algebra over the K theory of M.

The difference between the formula in 1. and that in the Atiyah-Singer Index Theorem is accounted for by the fact that for the twisted Spin complex $(E^{\pm}\otimes F, D_F^+)$,

$$ch\big(\sigma(E^{\pm}\otimes F, D_F^+)\big) = ch(E^{\pm}, D^+)\cdot ch(F)$$

and

$$Td(TM\otimes_{\mathbf{R}}\mathbf{C})\cdot \mathbf{ch}\big(\sigma(E^{\pm}, D^+)\big) = \widehat{\mathcal{A}}(TM).$$

For more on this see [ABP], [B], [G], [Gi], and [P].