

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 42 (1996)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: CHARACTERISTIC CLASSES, ELLIPTIC OPERATORS AND
COMPACT GROUP ACTIONS
Autor: HEITSCH, James L.
Kapitel: 2. The index of an elliptic complex
DOI: <https://doi.org/10.5169/seals-87879>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 14.03.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

2. THE INDEX OF AN ELLIPTIC COMPLEX

An *elliptic complex* (E, d) over a closed, oriented, n dimensional Riemannian manifold M consists of:

- a) a finite collection of finite dimensional complex vector bundles

$$E_0, E_1, \dots, E_k$$

- b) a collection of smooth differential operators

$$d_i : C^\infty(E_i) \rightarrow C^\infty(E_{i+1})$$

- c) The operators d_i are required to satisfy

$$d_{i+1} \cdot d_i = 0$$

and an additional technical condition called ellipticity. This condition makes possible the construction a virtual bundle, i.e. the formal difference of two vector bundles, over TM which carries a great deal of information about (E, d) . This virtual bundle $\sigma(E, d)$ is called the symbol of (E, d) and it defines a class $[\sigma(E, d)]$, also called the symbol, in the K theory with compact supports of TM .

EXAMPLES

1. The de Rham complex, where

$T_{\mathbb{C}}^*M$ = complexified cotangent bundle of M

$E_i = \Lambda^i T_{\mathbb{C}}^*M$ the i th exterior power of $T_{\mathbb{C}}^*M$

$C^\infty(E_i)$ = smooth complex i forms on M

d_i = the usual exterior derivative

2. The Dolbeault complex
 3. The Signature complex (see [AS])
 4. The twisted Spin complex.

SOME FACTS ABOUT ELLIPTIC COMPLEXES

Set $H^i(E, d) = \ker d_i / \text{image } d_{i-1}$. If M is compact, then $\dim H^i(E, d) < \infty$, and we may define

$$\text{Index}(E, d) = \sum_{i=0}^k (-1)^i \dim H^i(E, d).$$

This is a very important invariant. Special cases of (E, d) yield the

1. Euler class $\chi(M)$ of M (de Rham complex)
2. Signature of M (Signature complex)
3. Euler class $\chi(M, V)$ (Dolbeault complex)
4. \hat{A} genus of M (Spin complex).

The Atiyah-Singer Index Theorem tells how to compute this invariant from topological information about M and (E, d) . In particular, it says

THEOREM 2.1 ([AS]).

$$\text{Index}(E, d) = \int_M Td(TM \otimes_{\mathbf{R}} \mathbf{C}) \cdot \text{ch}(\sigma(E, d)).$$

The theorems quoted above are all special cases of this theorem. We now give an idea of how to prove this deep and important theorem.

On each E_i choose an Hermitian inner product denoted $(\ , \)_i$. This induces an inner product $\langle \ , \ \rangle_i$ on $C^\infty(E_i)$ by the formula

$$\langle s_1, s_2 \rangle_i = \int_M (s_1(x), s_2(x))_i dx.$$

Using $\langle \ , \ \rangle_i$ we define the adjoints

$$d_i^* : C^\infty(E_i) \rightarrow C^\infty(E_{i-1})$$

by

$$\langle s_1, d_i^* s_2 \rangle_{i-1} = \langle d_{i-1} s_1, s_2 \rangle_i$$

where

$$s_1 \in C^\infty(E_{i-1}), \quad s_2 \in C^\infty(E_i).$$

The Laplacian $\Delta_i : C^\infty(E_i) \rightarrow C^\infty(E_i)$ is defined by

$$\Delta_i = d_{i-1} d_i^* + d_{i+1}^* d_i,$$

$$(As)(x) = \int_M k(x, y)s(y)dy.$$

Note that $k(x, y)$ is a linear map from $E_{i,y}$, the fiber over y , to $E_{i,x}$, the fiber over x , so $k(x, x) : E_{i,x} \rightarrow E_{i,x}$ has a well defined trace. The section $k(x, y)$ is called the Schwartz kernel of A . Any smoothing operator on a compact manifold is of trace class and its trace is given by $\text{tr}(A) = \int_M \text{tr} k(x, x)dx$.

To see this for $e^{-t\Delta_i}$, note that its Schwartz kernel $k_t^i(x, y)$ must be given as follows: For each λ_j choose an orthonormal basis ϕ_j^v , $v = 1, \dots, \dim E_i(\lambda_j)$ of $E_i(\lambda_j)$. Then

$$k_t^i(x, y) = \sum_{j=0}^{\infty} e^{-t\lambda_j} \left[\sum_v \phi_j^v(x)\phi_j^v(y) \right].$$

Here $k_t^i(x, y) : E_{i,y} \rightarrow E_{i,x}$ acts on $w \in E_{i,y}$ by

$$k_t^i(x, y)w = \sum_{j=0}^{\infty} e^{-t\lambda_j} \left[\sum_v (\phi_j^v(y), w)_i \cdot \phi_j^v(x) \right].$$

The trace of $k_t^i(x, x)$ is then given by

$$\sum_{j=0}^{\infty} e^{-t\lambda_j} \left[\sum_v (\phi_j^v(x), \phi_j^v(x))_i \right]$$

and the result follows by integrating over M .

Now, since $e^{-t\lambda_0} = 1$ for all t , we have $e^{-t\lambda_0} \sum_{i=0}^k (-1)^i \dim E_i(\lambda_0) = \text{Index}(E, d)$, for all t . In addition $e^{-t\lambda_j} \sum_{i=0}^k (-1)^i \dim E_i(\lambda_j) = 0$ for $j > 0$, and for all t . Thus we have

THEOREM 2.3. *If M is compact, then for all $t > 0$,*

$$\begin{aligned} \text{Index}(E, d) &= \sum_{j=0}^{\infty} \left[\sum_{i=0}^k (-1)^i e^{-t\lambda_j} \dim E_i(\lambda_j) \right] \\ &= \sum_{i=0}^k \left[\sum_{j=0}^{\infty} (-1)^i e^{-t\lambda_j} \dim E_i(\lambda_j) \right] \\ &= \sum_{i=0}^k (-1)^i \text{tr} e^{-t\Delta_i}. \end{aligned}$$

The Index Theorem now follows from two other results.

- 1) Set $k_t(x) = \sum_{i=0}^k (-1)^i \text{tr } k_t^i(x, x)$. Then for t near 0, $k_t(x)$ has an asymptotic expansion of the form

$$k_t(x) = \sum_{j \geq -n} t^{j/2} a_j(x).$$

As $\int_M k_t(x) dx = \sum_{i=0}^k (-1)^i \text{tr } e^{-t\Delta_i} = \text{Index}(E, d)$ is independent of t , we have

$$\text{Index}(E, d) = \int_M a_0(x) dx.$$

Now, for any twisted Dirac operator D_F^+ , one can prove that the differential n form $a_0(x) dx$ is the degree n part of the form constructed from the connections on TM and F which represents $\widehat{A}(TM) \cdot \text{ch}(F)$. Thus, we have

$$\text{Index}(D_F^+) = \int_M \widehat{A}(TM) \cdot \text{ch}(F).$$

- 2) $\text{Index}(E, d)$ depends only on the K theory class $[\sigma(E, d)]$. Given this and the formula above for $\text{Index}(D_F^+)$, one may use well known arguments in K theory to extend the result in 1. to all elliptic complexes. The essential fact is that the symbols of twisted Dirac operators generate the K theory with compact supports of TM as an algebra over the K theory of M .

The difference between the formula in 1. and that in the Atiyah-Singer Index Theorem is accounted for by the fact that for the twisted Spin complex $(E^\pm \otimes F, D_F^+)$,

$$\text{ch}(\sigma(E^\pm \otimes F, D_F^+)) = \text{ch}(E^\pm, D^+) \cdot \text{ch}(F)$$

and

$$\text{Td}(TM \otimes_{\mathbf{R}} \mathbf{C}) \cdot \mathbf{ch}(\sigma(E^\pm, D^+)) = \widehat{A}(TM).$$

For more on this see [ABP], [B], [G], [Gi], and [P].