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numbers. For another application of Haefliger forms and their cohomology, see [He].

An immediate corollary is the following theorem of Atiyah and Hirzebruch.

THEOREM 5.3 ([AH]). *Let M be a compact connected, oriented manifold which admits a Spin structure. If a compact connected Lie group acts non-trivially on M, then the $\widehat{A}$ genus of M is zero.*

Theorem 5.2 is an application of the Lefschetz fixed point theorem for complexes elliptic along the leaves of a foliated manifold. We explain the classical Lefschetz theorem for elliptic complexes and give an outline of how to prove it. The original proofs of this theorem relied on the fact that the underlying manifold was compact. We outline a proof which does not rely on that fact, and so can be generalized to complexes defined along the leaves of a compact foliated manifold. Note that such leaves are in general not compact, but the fact that they come from a foliation of a compact manifold means that they have uniformly bounded geometry. It is this property which allows us to prove the foliation version of the Lefschetz theorem. We then show how the Lefschetz theorem leads to Theorem 5.2. Finally, we give a brief explanation of a very general rigidity theorem conjectured by Witten and proven by Bott and Taubes.

This paper is based on lectures given at the conference Actions Différentiables de Groupes Compacts, Espaces d'Orbites et Classes Caractéristiques, held at the Université des Sciences et Techniques du Languedoc in Montpellier in January, 1994. The author wishes to thank the organizers, especially Daniel Lehmann and Pierre Molino, for extending the invitation to him to speak at the conference and for making his stay in Montpellier so pleasant.

1. CHARACTERISTIC CLASSES AND MULTIPLICATIVE SEQUENCES

All objects considered in this paper will be smooth. Let E be an n dimensional complex vector bundle over the real manifold M. Denote the space of smooth sections of E by $C^\infty(E)$. A connection on E is a linear map $\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E)$ satisfying

$$\nabla(f \cdot \sigma) = df \otimes \sigma + f \cdot \nabla\sigma$$

for any $\sigma \in C^\infty(E)$ and $f \in C^\infty(M)$, the smooth functions on M. T^*M denotes the cotangent bundle of M.

If $\sigma_1, \dots, \sigma_n$ is a local basis of $C^\infty(E)$ on an open set U, the local connection form θ_U (which is an $n \times n$ matrix of 1 forms) is defined by

$$\nabla \sigma_i = \sum_{j=1}^{n} \theta^i_{U,j} \otimes \sigma_j .$$

The local curvature form Ω_U is the $n \times n$ matrix of 2 forms

$$\Omega_U = d\theta_U - \theta_U \wedge \theta_U .$$

It is not difficult to show that if $\tau_1, \dots, \tau_n$ is another local basis on the open set V with $\tau_i = \sum_{j=1}^{n} g_{ij}\sigma_j$ on $U \cap V$, with $g_{ij} \in C^\infty(U \cap V)$, then on $U \cap V$

$$\Omega_U = g\Omega_V g^{-1} \tag{1.1}$$

where $g = [g_{ij}]$.

Now consider the local differential form on U, $\det\left(I - \frac{1}{2\pi i}\Omega_U\right)$. Because of (1.1), this is actually a well defined *global* form on M. This form depends only on ∇ and it is closed, so it defines a cohomology class $c(E)$, the total Chern class of E, which actually takes values in the real de Rham cohomology of M. This class depends only on E and we may write

$$c(E) = 1 + c_1(E) + \dots + c_n(E)$$

where $c_k(E) \in H^{2k}(M, \mathbf{R})$ is the kth Chern class of E.

If E is an n dimensional real vector bundle over M, it is easy to show that $c_{2k+1}(E \otimes_{\mathbf{R}} \mathbf{C}) = 0$, and the kth Pontrjagin class of E is defined to be

$$p_k(E) = (-1)^k c_{2k}(E \otimes_{\mathbf{R}} \mathbf{C}) .$$

For more on this see [KN] and [M].

Let $Q(z) = \sum_{i=0}^{\infty} b_i z^i$ be a formal power series in z. Associated to Q is the *multiplicative sequence* $K = (K_0, K_1, K_2, \dots)$ where each K_j is a polynomial in j indeterminants, $K_j(\sigma_1, \dots, \sigma_j)$ given as follows. Denote by Q_j the degree j part of $Q(z_1)\dots Q(z_j)$, where each z_i has degree 1. Q_j is a symmetric polynomial in the z_i so it can be written as a polynomial in the elementary symmetric polynomials $\sigma_1, \dots, \sigma_j$ in $z_1, \dots, z_j$, i.e.

$$Q_j = K_j(\sigma_1, \dots, \sigma_j) .$$

For example, if $Q(z) = 1+z$, then $Q_j = z_1 \dots z_j = \sigma_j$ and $K_j(\sigma_1, \dots, \sigma_j) = \sigma_j$.

If $Q(z)$ is an even power series, $Q(z) = \sum_{i=0}^{\infty} b_{2i} z^{2i}$, then the degree $2j$ part of $Q(z_1)\dots Q(z_j)$ can be written as a polynomial in the elementary symmetric polynomials $\gamma_1, \dots, \gamma_j$ in $z_1^2, \dots, z_j^2$. We set $K_j(\gamma_1, \dots, \gamma_j)$ to be this polynomial.

DEFINITION 1.2. *(a) Let E be an n dimensional complex vector bundle over M and $Q(z)$ a formal power series with associated multiplicative sequence $K = (K_0, K_1, \ldots)$. The K genus of E, $K(E)$ is the de Rham cohomology class*

$$K(E) = \sum_{j=0}^{\infty} K_j\big(c_1(E), \ldots, c_j(E)\big).$$

(as $K_j\big(c_1(E), \ldots, c_j(E)\big) \in H^{2j}(M, \mathbf{R})$, this is actually a finite sum).

(b) Let E be an n dimensional real vector bundle over M and $Q(z)$ an even formal power series with associated multiplicative sequence $K = (K_0, K_1, \ldots)$. Then the K genus of E is $K(E) = \sum\limits_{j=0}^{\infty} K_j\big(p_1(E), \ldots, p_j(E)\big)$.

K is called a multiplicative sequence because $K(E_1 \oplus E_2) = K(E_1) \cdot K(E_2)$.

IMPORTANT EXAMPLES

1. $Q(z) = 1 + z$. Then

$$K(z) = 1 + c_1(E) + c_2(E) + \cdots = c(E).$$

2. $Q(z) = z/\tanh(z)$, which is even and gives the L genus of Hirzebruch.

Recall that the signature Sign(M) of a compact oriented $4k$ dimensional manifold M is the signature of the quadratic form on $H^{2k}(M, \mathbf{R})$ given by $\alpha, \beta \longmapsto \int_M \alpha \cdot \beta$.

THEOREM 1.3 (Hirzebruch [H]).

$$\mathrm{Sign}(M) = \int_M L_k\big(p_1(TM), \ldots, p_k(TM)\big),$$

where TM is the tangent bundle of M.

Thus Sign(M) is completely determined by the Pontrjagin classes of M.

3. $Q(z) = z/(1 - e^{-z})$, gives the Todd genus Td.

Let M be a compact, complex n dimensional manifold and V a holomorphic vector bundle over M. Recall the Dolbeault complex of V

$$0 \to A^{0,0}(V) \xrightarrow{\overline{\partial}_0} A^{0,1}(V) \xrightarrow{\overline{\partial}_1} \cdots \xrightarrow{\overline{\partial}_{n-1}} A^{0,n}(V) \to 0$$

where $A^{0,q}(V)$ is the space of differential forms on M of type $0,q$ with coefficients in V. $H^q(M,V) =$ kernel $\overline{\partial}_q/$ image $\overline{\partial}_{q-1}$ and it is finite dimensional (and isomorphic to $H^q\big(M,\mathcal{O}(V)\big)$ where $\mathcal{O}(V)$ is the sheaf of germs of holomorphic sections of V).

A fundamental invariant of V is its Euler class

$$\chi(M,V) = \sum_{q=0}^{n}(-1)^q \dim H^q(M,V)\,.$$

The Riemann-Roch problem is to calculate this integer from topological information about M and V. The solution is given as follows. Suppose dimension $V = k$. $e^{z_1} + \cdots + e^{z_k}$ is symmetric in the z_i so may be written as a power series in $\sigma_1, \ldots, \sigma_k$, i.e. $\sum_{i=1}^{k} e^{z_i} = k + \mathrm{ch}_1(\sigma_1) + \mathrm{ch}_2(\sigma_1,\sigma_2) + \cdots$ where

$$\mathrm{ch}_j\big(\sigma_1(z_1,\ldots,z_j),\ldots,\sigma_j(z_1,\ldots,z_j)\big) = \sum_{i=1}^{k} z_i^j/j!\,.$$

Set $\mathrm{ch}(V) = k + \mathrm{ch}_1\big(c_1(V)\big) + \mathrm{ch}_2\big(c_1(V),c_2(V)\big) + \cdots$.

THEOREM 1.4 (The Riemann-Roch Theorem, [AS]).

$$\chi(M,V) = \int_M Td(TM)\cdot \mathrm{ch}(V)\,.$$

Thus $\chi(M,V)$ is completely determined by the Chern classes of M and V.

4. $Q(z) = (z/2)/\sinh(z/2) = z/(e^{z/2} - e^{-z/2})$ is an even function and gives the $\widehat{\mathcal{A}}$ genus.

Recall that Spin(n) is the simply connected double cover of $SO(n)$. A Spin structure on an oriented Riemannian manifold M of dimension n is a principal Spin(n) bundle P over M and an isomorphism of oriented bundles

$$P \times_{\mathrm{Spin(n)}} \mathbf{R^n} \simeq TM.$$

Spin(n) has a complex representation space Δ of dimension 2^n. See [ABS], [LM]. If $n = 2k$, Δ may be written as $\Delta = \Delta^+ \oplus \Delta^-$ where $\Delta^\pm$ are irreducible representations of dimension 2^{n-1}. Set $E^\pm = P \times_{\mathrm{Spin}(n)} \Delta^\pm$. The metric connection on M defines one on $E = E^+ \oplus E^-$, denoted ∇. The Dirac operator $D^+ : C^\infty(E^+) \to C^\infty(E^-)$ is defined as follows. Let $c : C^\infty\big(T^*M \otimes E\big) \to C^\infty(E)$ be Clifford multiplication (we identify T^*M with TM using the metric on M). Then $D = c\cdot\nabla : C^\infty(E) \to C^\infty(E)$ and

D maps $C^\infty(E^+)$ to $C^\infty(E^-)$ and vice-versa, since c does. Thus we may write

$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix} \tag{1.5}$$

where $D^\pm : C^\infty(E^\pm) \to C^\infty(E^\mp)$. (See [AS] or [LM]). Now $\ker D^+$ and $\operatorname{coker} D^+ (\simeq \ker D^-)$ are finite dimensional and the Spinor index of M,

$$\operatorname{Spin}(M) = \dim \ker D^+ - \dim \operatorname{coker} D^+ .$$

THEOREM 1.6 ([AS]). *If M is a Spin manifold of* $\dim 2k$ *then*

$$\operatorname{Spin}(M) = \int_M \widehat{A}(TM) .$$

In particular, if $2k \equiv 2(4)$, then Spin(M) $= 0$ as $\widehat{A}(TM)$ involves only the Pontrjagin classes of M and these occur only in dimensions $\equiv 0(4)$.

More generally, we may construct the twisted spinor complex. For this, let F be a complex bundle over M with hermitian metric and connection. Combining the connection on E with that on F we obtain a connection on $E \otimes F$. Composing this with Clifford multiplication

$$c : C^\infty(T^*M \otimes E \otimes F) \to C^\infty(E \otimes F)$$

we obtain the twisted Dirac operator D_F on $E \otimes F$. As before D_F interchanges $E^+ \otimes F$ and $E^- \otimes F$ and we get the *twisted Spin complex*

$$0 \to C^\infty(E^+ \otimes F) \xrightarrow{D_F^+} C^\infty(E^- \otimes F) \to 0 .$$

The kernel and cokernel of D_F^+ are finite dimensional and the twisted spinor index is

$$\operatorname{Spin}(M, F) = \dim \ker D_F^+ - \dim \operatorname{coker} D_F^+ .$$

THEOREM 1.7 ([AS]).

$$\operatorname{Spin}(M, F) = \int_M \widehat{A}(TM) \cdot \operatorname{ch}(F) .$$