

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 41 (1995)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** HIGHER EULER CHARACTERISTICS (I)  
**Autor:** Geoghegan, Ross / Nicas, Andrew  
**Kapitel:** Introduction  
**DOI:** <https://doi.org/10.5169/seals-61816>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 19.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## HIGHER EULER CHARACTERISTICS (I)

by Ross GEOGHEGAN<sup>1)</sup> and Andrew NICAS<sup>2)</sup>

*To Peter Hilton on the occasion of his 70-th birthday.*

ABSTRACT. The classical Euler characteristic  $\chi \equiv \chi_0$  of a finite complex lies at the bottom of a sequence of homotopy invariants. The next invariant in this sequence  $\chi_1$  is introduced here and studied in some detail. The rest of the sequence,  $\chi_n$  with  $n \geq 2$ , will be discussed in a sequel paper. Applications to geometric group theory are found by considering the behavior of  $\chi_1$  on an aspherical finite complex of fundamental group  $G$ . Just as the  $\chi(G) \neq 0$  implies that the center of  $G$  is trivial (Gottlieb's Theorem), it is shown here that (under a weak additional hypothesis and using rational coefficients)  $\chi_1(G) \neq 0$  implies that the center of  $G$  is infinite cyclic. We also find a generalization of Gottlieb's Theorem in which the Lefschetz number of an automorphism of  $G$  is related to the fixed subgroup of the automorphism.

### INTRODUCTION

From our point of view, the classical Euler characteristic of a finite complex is “zero-th order”. In this paper we introduce a “first order” analog, a new invariant in topology and group theory. In a sequel paper and in [GNO] we extend these ideas to an “ $n$ -th order” Euler characteristic for all positive  $n$ .

For a finite complex  $X$ , the new invariant  $\chi_1(X; R)$ , defined in §1, comes in different forms, depending on the coefficient ring  $R$ ; and a more sophisticated version  $\tilde{\chi}_1(X; R)$  defined in §2, involves the universal cover of  $X$ . By contrast, the classical analogs of these are essentially the same, namely the integer  $\chi(X)$ . We should tell the reader from the start that all our first order invariants are trivial if  $X$  is simply connected.

---

1991 *Mathematics Subject Classification*. Primary 55M20; Secondary 19D55, 20F28, 20F32.

<sup>1)</sup> Partially supported by the National Science Foundation.

<sup>2)</sup> Partially supported by the Natural Sciences and Engineering Research Council of Canada.

The authors acknowledge the hospitality of the Institute for Advanced Study and support from NSF grant DMS 9304580.

The paper begins with three rather different definitions of  $\chi_1(X; R)$ , a discussion of their equivalence, and some motivation for these definitions. Our point of view is geometric, but for readers more interested in homotopy theory we include (at the end of §1) a brief discussion of a fourth definition in terms of stable homotopy theory.

Next, we discuss the computation of  $\chi_1(X; R)$  for 1-complexes, certain 2-complexes, 3-dimensional lens spaces, circle bundles and mapping tori.

In §5 and §7, we apply these ideas to group theory. Motivated by Gottlieb's theorem [Got] that if  $X$  is a finite aspherical complex with fundamental group  $G$  and if  $\chi(G) \equiv \chi(X) \neq 0$  then the center of  $G$  is trivial, we find an analog (Theorem 5.4) which says, roughly, that if  $\chi_1(G; \mathbf{Q}) \equiv \chi_1(X; \mathbf{Q}) \neq 0$  then the center of  $G$  is infinite cyclic. This leads us to surprising generalization of Gottlieb's theorem (Theorem 8.1). In this theorem, one is given an automorphism  $\theta$  of  $G$  induced by a map  $f: X \rightarrow X$ . By the *Lefschetz number*,  $L(\theta)$ , of  $\theta$  we mean the Lefschetz number of  $f$ . We prove (under a weak  $K$ -theoretic hypothesis on  $G$ ) that if  $\theta$  has order  $r$  in the group of outer automorphisms of  $G$  and if  $\sum_{i=0}^{r-1} L(\theta^i) \neq 0$  then the intersection of the center of  $G$  and the fixed subgroup of  $\theta$  is trivial. We do not know of a previous theorem which relates so directly the fixed subgroup of an automorphism to the classical fixed point theory of the associated map.

We also introduce a more refined invariant  $\tilde{\chi}_1(X) \in H^1(\Gamma, HH_1(\mathbf{Z}G))$  where  $HH_1(\mathbf{Z}G)$  is the first Hochschild homology group of  $\mathbf{Z}G$  (see §1). This is an analog of what one obtains when one computes the classical Euler characteristic as a Hattori-Stallings trace in the universal cover of  $X$ . In the classical case one essentially recovers  $\chi(X)$ , but a significant distinction appears in the case of the "higher order" invariants. In a natural manner,  $\tilde{\chi}_1(X)$  maps to  $\chi_1(X)$  regarded as an element of  $H^1(\Gamma, H_1(G))$ . Applications of  $\tilde{\chi}_1$  to characteristic classes and Seifert fiber spaces will be given in [GN<sub>5</sub>].

The ideas presented here are an outgrowth of the one-parameter fixed point theory developed in [GN<sub>1</sub>] and its application to dynamics in [GN<sub>2</sub>]. A summary has appeared as [GN<sub>3</sub>]. Most of this paper can be read independently of [GN<sub>1</sub>] and [GN<sub>2</sub>]. We make modest use of a few technical propositions from [GN<sub>1</sub>] in §2 and §3; in §10 a difficult result from [GN<sub>1</sub>] is invoked.

One of the definitions of  $\chi_1(X; R)$  employs a formula introduced more than twenty years ago in [Kn]; we thank Boris Okun for drawing our attention to that paper.