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Autor: Cairns, G. / Davis, G. / Elton, D. / Kolganova, A. / Perversi, P.
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Notice that in condition (b) of the above definition, finite orbits for general group actions are a direct generalization of periodic orbits for $\mathbf{Z}$-actions. Phenomena very similar to that of chaotic actions have been studied for decades, though the word "chaos" was not used. For example, as P. Eberlein relates on his description [E] of the work on the geodesic flow in the 1920's. "The object of most of the works in this period was to establish topological dynamical properties of the geodesic flow such as the density of periodic trajectories (= closed geodesics) and the existence of a dense trajectory (topological transitivity)." Now it can be easily verified (cf. [BBCDS], [GW] and [Si]) that a chaotic action of a group G on an infinite metric space M is "chaotic" in the popular sense that it has sensitive dependence on initial conditions; that is, there exists $\delta > 0$ such that for every open set U in M there exist, $x, y \in U$ and $g \in G$ such that are $g(x)$ and $g(y)$ are at least distance δ apart.

The basic example of a chaotic action is provided by the linear action of $SL(n, \mathbf{Z})$ on the torus $\mathbf{T}^n$, for any $n \geqslant 2$. Condition (b) in the above definition is verified for this action, since the points with rational coordinates have finite orbit. To see that condition (a) is satisfied, one shows that every invariant open subset of $\mathbf{T}^n$ is dense.

In this paper, we provide a collection of observations and questions concerning chaotic group actions. These actions are not merely an artificial generalization of chaotic $\mathbf{Z}$ actions; as we show in Section 3 below, there exist chaotic actions of a group G for which the restriction to every one generator subgroup of G is not chaotic. In our study we do not assume any differentiability or measure theoretic hypotheses and so our results are all quite elementary. Nevertheless, as we hope to convince the reader, the structure is sufficiently rich as to provide a variety of results.

2. Chaos Equals Residual Finiteness

Now the two conditions in the above definition of a chaotic action are quite different in nature. The first condition is an irreducibility condition. The second condition is just a disguised form of residual finiteness. Indeed, recall that a group G is said to be residually finite if for every non-identity element g of G, there is a normal subgroup, not containing g, of finite index in G. Then one has:

THEOREM 1. *For a group G, the following conditions are equivalent:*
(a) G is residually finite,

(b) *there is a faithful action of* G *with finite orbits dense on some Hausdorff topological space* M,

(c) *there is a faithful action of* G *with all orbits finite on some Hausdorff topological space* M,

(d) *there is a faithful chaotic action of* G *on some Hausdorff topological space* M.

Proof. The proof is particularly simple. Clearly (d) and (c) each imply (b). We show that (b) implies (a) and that (a) implies (c) and (d).

(b $\Rightarrow$ a). Suppose that a group G acts faithfully with finite orbits dense on a space M and that g is an element of G, other than the identity element. Since the set F of points of M with finite orbit under G is dense in M, there exists a point $x \in F$ which is not fixed by g. Let $O(x)$ denote the orbit of x under G and let H_x denote the subgroup of G that fixes $O(x)$ point by point. Then clearly H_x is the required normal subgroup of finite index.

(a $\Rightarrow$ c). If G is residually finite, then for each non-identity element $g \in G$, there is a normal finite index subgroup H_g that doesn't contain g. Let M_g denote the quotient space G/H_g. Then G acts on M_g by left translation. Now let M be the disjoint union $\coprod_{g \neq \mathrm{id}} M_g$, equipped with the discrete topology. Then G acts faithfully on M and every point has finite orbit, by construction.

(a $\Rightarrow$ d). If G is finite, then G acts chaotically on itself, with the discrete topology. Suppose that G is infinite and consider the compact product space $\{0, 1\}^G$. The natural action of G on $\{0, 1\}^G$ is given by

$$g(f)(x) = f(g^{-1}x) ,$$

for all $g, x \in G$ and $f: G \to \{0, 1\}$. It is an elementary exercise to show that this action is transitive. Now suppose that G is residually finite. Let $\{x_1, \ldots, x_n\}$ be a finite set of distinct elements of G, choose numbers $y_1, \ldots, y_n \in \{0, 1\}$ and consider the open set U of functions $f: G \to \{0, 1\}$ for which $f(x_i) = y_i$ for all i. We will show that U contains an element with finite orbit under G. Notice that since G is residually finite, for every pair of distinct elements $a, b \in G$, there exists a finite index normal subgroup H of G such that a and b belong to different cosets of H. It follows that there exists a finite index normal subgroup K of G such that the elements x_i belong to pairwise distinct cosets of K. Now let f be any function which is constant on the cosets of K and which takes the value y_i on the coset containing x_i. So $f \in U$ and clearly f has finite orbit under G. $\square$

The above theorem is quite useful. For instance, it shows that groups which act chaotically and faithfully have no infinite simple subgroups and so, in particular, they cannot themselves be infinite simple groups. In another direction, the group $\mathbf{Q}$ of rational numbers is not residually finite (see for example [We]) and so $\mathbf{Q}$ cannot act chaotically and faithfully. Other useful well known properties of residually finite groups (see for example [LS] and [MKS]) include: Finitely generated residually finite groups are Hopfian; that is, they are not isomorphic to any of their proper quotient groups. Finitely generated residually finite groups have residually finite automorphism groups. Finitely presented residually finite groups have solvable word problem.

In passing, let us remark that the above theorem provides an elementary and unified manner to prove residual finiteness in many cases:

COROLLARY 1. *The following groups are residually finite:*

(a) the matrix groups $SL(n, \mathbf{Z})$, *for all* $n > 1$,

(b) countably generated free groups,

(c) quotients of residually finite groups by finite normal subgroups,

(d) subgroups of residually finite groups,

(e) (finite and infinite) direct products of residually finite groups,

(f) wreath products of residually finite groups by finite groups.

Proof of Corollary 1. The statements in Corollary 1 are well known (though we haven't seen (e) in the literature). Part (a) uses the fact that $SL(n, \mathbf{Z})$ acts with finite orbits dense on $\mathbf{T}^n$, as mentioned in the introduction. Parts (c) and (d) follow immediately from the theorem; indeed, it is clear that if a group G acts faithfully with all orbits finite on a Hausdorff space M, then every subgroup of G also acts faithfully with all orbits finite on M. And if H is a finite normal subgroup, then G/H acts faithfully with all orbits finite on the Hausdorff orbit space M/H. To see Part (b), one first recalls that the free group on two generators is a subgroup of $SL(2, \mathbf{Z})$; indeed, by Sanov's theorem (see [LS]), the subgroup of $SL(2, \mathbf{Z})$ generated by the matrices

$$\alpha = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

is isomorphic to the free group $\mathbf{Z} * \mathbf{Z}$. So by Part (d), $\mathbf{Z} * \mathbf{Z}$ is residually finite. Part (b) then follows from Part (d) again, using the fact that every countably generated free group is a subgroup of $\mathbf{Z} * \mathbf{Z}$. Part (e) is proved as follows:

suppose that groups $G_i, i \in I$ act faithfully with all orbits finite on the Hausdorff spaces $M_i, i \in I$ respectively. Now to each space M_i, add an additional isolated element x_i and denote $\tilde{M}_i$ the union $M_i \cup \{x_i\}$. Then we define an action of G_i on $\tilde{M}_i$ by using the action of G_i on M_i and making x_i a fixed point. Clearly G_i acts faithfully with all orbits finite on $\tilde{M}_i$. Now let M denote the subset of the infinite product $\prod_{i \in I} \tilde{M}_i$ composed of all elements $(y_i)_{i \in I}$ for which only finitely many of the y_i are different from x_i. We equip M with the topology induced by the product topology. Then clearly the infinite direct product $\prod_{i \in I} G_i$ acts faithfully with all orbits finite on M.

Finally Part (f) is similar to Part (e); suppose that a group G acts chaotically on a space M and that H is a finite group. Then there is a natural action of the wreath product $G\mathrm{Wr}H$ on the space $M \times H$, where H is given the discrete topology (see [H]). It is easy to see that this action is chaotic. □

The following groups are known to be residually finite: Fuchsian groups [LS], the mapping class groups of compact Riemann surfaces [G], arithmetic groups [Se] and the group of p-adic integers [We]. It would be interesting to find natural chaotic actions of these groups.

3. Constructions of Chaotic Group Actions

First recall that there are many examples of chaotic $\mathbf{Z}$-actions; that is, chaotic homeomorphisms. Perhaps the most basic example is that of the Anosov diffeomorphisms of tori and infranilmanifolds (see [Sm], [Mann]); these maps are chaotic since their periodic points are dense [BR] and by Anosov's closing lemma (see for instance [Sh]), they are transitive on their nonwandering set. (The Anosov diffeomorphisms of tori are just the linear hyperbolic maps; that is, linear maps with no eigenvalues on the unit circle.) Similarly, the pseudo-Anosov maps of compact surfaces are also chaotic (see Exposé 9 in [FLP] and the diagrams in [Mañ], pages 111-116).

Let us now give some general results.

Theorem 2. *Consider a Hausdorff space* M *and the group* $\mathrm{Hom}(M)$ *of homeomorphisms of* M. *Then one has:*

(a) If there are group inclusions

$$G \leqslant H \leqslant K \leqslant \mathrm{Hom}(M)$$

then the action of H *on* M *is chaotic if the actions of* G *and* K *on* M *are chaotic.*