Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 41 (1995)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: CHAOTIC GROUP ACTIONS
Autor: Cairns, G. / Davis, G. / Elton, D. / Kolganova, A. / Perversi, P.
Kapitel: 1. Introduction
DOI: https://doi.org/10.5169/seals-61820

L'Enseignement Mathématique, t. 41 (1995), p. 123-133

CHAOTIC GROUP ACTIONS

by G. CAIRNS, G. DAVIS, D. ELTON, A. KOLGANOVA and P. PERVERSI

ABSTRACT. We introduce the notion of chaotic group actions and give a preliminary report on their properties. In particular, we show that a group G possesses a faithful chaotic action on some Hausdorff space if and only if G is residually finite. This gives an elementary unified proof of the residual finiteness of certain groups. We also show that the circle does not admit a chaotic action of any group, whilst every smooth compact surface admits a chaotic $\mathbf{Z}$-action.

1. INTRODUCTION

In recent years an enormous amount of work has been conducted on chaotic dynamical systems. Most of this work has been concerned with the iteration of single maps; in other words, with group (or semi-group) actions of the additive group $\mathbf{Z}$. Now, according to R. Devaney's [D2] definition (see also [BBCDS], [GW] and [Si]), a map is chaotic if it is topologically transitive and if the set of periodic points is dense. The purpose of this present paper is to introduce the analogous notion for actions of arbitrary groups:

Definition. Suppose that a group G acts continuously on a Hausdorff topological space M. Then we say that the action of G on M is *chaotic* if the following two conditions are met:

(a) *topological transitivity:* for every pair of non-empty open subsets U and V of M, there is an element $g \in G$ such that $g(U) \cap V \neq \varnothing$.

(b) *finite orbits dense:* the set of points in M whose orbit under G is finite is a dense subset of M.

1991 *Mathematics Subject Classification.* Primary 58F08 Secondary 28D05.
Key words and phrases. Group action, chaos.
Acknowledgement. We would like to thank John Banks, Swarup Gadde and Peter Stacey for their helpful comments. We also acknowledge an unknown referee of an earlier version of this paper who provided the construction showing that residually finite groups possess chaotic actions.

Notice that in condition (b) of the above definition, finite orbits for general group actions are a direct generalization of periodic orbits for $\mathbf{Z}$-actions. Phenomena very similar to that of chaotic actions have been studied for decades, though the word "chaos" was not used. For example, as P. Eberlein relates on his description [E] of the work on the geodesic flow in the 1920's. "The object of most of the works in this period was to establish topological dynamical properties of the geodesic flow such as the density of periodic trajectories (= closed geodesics) and the existence of a dense trajectory (topological transitivity)." Now it can be easily verified (cf. [BBCDS], [GW] and [Si]) that a chaotic action of a group G on an infinite metric space M is "chaotic" in the popular sense that it has sensitive dependence on initial conditions; that is, there exists $\delta > 0$ such that for every open set U in M there exist, $x, y \in U$ and $g \in G$ such that are $g(x)$ and $g(y)$ are at least distance δ apart.

The basic example of a chaotic action is provided by the linear action of $SL(n, \mathbf{Z})$ on the torus $\mathbf{T}^n$, for any $n \geqslant 2$. Condition (b) in the above definition is verified for this action, since the points with rational coordinates have finite orbit. To see that condition (a) is satisfied, one shows that every invariant open subset of $\mathbf{T}^n$ is dense.

In this paper, we provide a collection of observations and questions concerning chaotic group actions. These actions are not merely an artificial generalization of chaotic $\mathbf{Z}$ actions; as we show in Section 3 below, there exist chaotic actions of a group G for which the restriction to every one generator subgroup of G is not chaotic. In our study we do not assume any differentiability or measure theoretic hypotheses and so our results are all quite elementary. Nevertheless, as we hope to convince the reader, the structure is sufficiently rich as to provide a variety of results.

2. Chaos Equals Residual Finiteness

Now the two conditions in the above definition of a chaotic action are quite different in nature. The first condition is an irreducibility condition. The second condition is just a disguised form of residual finiteness. Indeed, recall that a group G is said to be residually finite if for every non-identity element g of G, there is a normal subgroup, not containing g, of finite index in G. Then one has:

THEOREM 1. *For a group G, the following conditions are equivalent:*

(a) G is residually finite,