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Artikel: CONCERNING A REAL-VALUED CONTINUOUS FUNCTION ON THE INTERVAL WITH GRAPH OF HAUSDORFF DIMENSION 2

Autor: Wingren, Peter

Kapitel: 5. The main result

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Mass Distribution Principle. Let μ *be a mass distribution on* $A \subset \mathbf{R}^n$. *If there exist constants* $c > 0$ *and* $\delta > 0$ *such that, for all dyadic cubes* $Q \subset \mathbf{R}^n$ *with* $|Q| \leqslant \delta$,

$$\mu(Q) \leqslant c \cdot |Q|^{\alpha}, \tag{10}$$

then

$$\alpha \leqslant \dim_H(A). \tag{11}$$

Proof. Let $\{Q_i\}_{i=1}^{\infty}$ be a covering of A with dyadic cubes of diameter not exceeding δ. Then

$$0 < \mu(A) \leqslant \mu\left(\bigcup_{i=1}^{\infty} Q_i\right) \leqslant \sum_{1}^{\infty} \mu(Q_i) \leqslant c \cdot \sum_{1}^{\infty} |Q_i|^{\alpha} \tag{12}$$

and hence the discontinuity in the $M^{\alpha}(A)$-graph from $+\infty$ to 0 occurs at a value not less than α. Thus

$$\alpha \leqslant \dim_H(A). \tag{13}$$

5. The main result

The notation used in the following theorem and in its proof can be found in Section 0.

THEOREM. *Let*

$$f(x) = \sum_{p=0}^{\infty} 2^{-p} \operatorname{dist}(2^{2^p} x, \mathbf{Z}), \quad x \in [0, 1]. \tag{14}$$

Then for every Borel subset B *of* $\operatorname{graph}(f)$ *with* $m(\operatorname{Proj}(B)) > 0$,

$$\dim_H(B) = 2. \tag{15}$$

Proof. Assume that B is a Borel set as above. From $\operatorname{graph}(f) \subset \mathbf{R}^2$ there follows

$$\dim_H(B) \leqslant 2. \tag{16}$$

It will suffice to prove that

$$\alpha \leqslant \dim_H(B) \tag{17}$$

for an arbitrary positive $\alpha < 2$. Distribute the unit mass as in Lemma 1. Let Q be a dyadic cube with side length less than $\frac{1}{4}$. Then the side length of Q is 2^{-n} for some positive integer n and there is a positive integer p such that

$$2^{-2p+1} \leqslant 2^{-n} < 2^{-2p} . \tag{18}$$

Let D_0 be the smallest vertical band which inscribes Q, and so it has band width 2^{-n}. From the second inequality in (18) we conclude that D_0 is contained in a band from generation p. In the discussion and in the estimations which follow, just those bands which are of generations $p, p+1$ and $p+2$ play a role. We let D and D_L denote an arbitrary band from generation $p+1$ and its left half, respectively. On D_L we study $f(x)$ as a sum of two terms,

$$f(x) = \sum_{0}^{p+1} 2^{-k}\operatorname{dist}(2^{2^k}x, \mathbf{Z}) + \sum_{p+2}^{\infty} 2^{-k}\operatorname{dist}(2^{2^k}x, \mathbf{Z}) . \tag{19}$$

The first term is linear and the second periodic (one cycle on each subband from generation $p+2$). This implies that the distribution of mass via (5) on each $(p+2)$-subband (of D_L) is the same but translated a fixed distance d_D in y-direction. Now let D' be a $(p+2)$-subband of D_L and define

$$G_{D'}(y) := \mu(\{(x_1, x_2) \in D' \text{ and } x_2 \leqslant y\}) . \tag{20}$$

Then its derivative $g(y)$ exists a.e. and

$$\| g \| = 2^{-2p+2} . \tag{21}$$

If D' and D'' are neighbouring $(p+2)$-generation subbands of D_L, then $G_{D''}(y)$ is a translation of $G_{D'}(y)$ by d_D. Hence, we may use just one function G and its translates. The fixed translation d_D of mass in y-direction from one band to the next may be estimated by the derivative of the first sum of (19),

$$\begin{aligned} d_D &= 2^{-2p+2} \times \left| \frac{d}{dx}\left(\sum_{p=0}^{p+1} 2^{-k}\operatorname{dist}(2^{2^k}x, \mathbf{Z})\right) \right| \\ &\geqslant 2^{-2p+2}(2^{-(p+1)+2^{p+1}} - 2^{-p+2^p} - \ldots - 2) > 2^{-2p+1-(p+2)} . \end{aligned} \tag{22}$$

The last inequality holds for $p > 1$, because the rapid decrease of the successive terms in the parenthesis implies that its value is larger than half the first term. (It is easy to check that this estimation also works for $(p+2)$-generation bands in the right band half D_R of a $(p+1)$-generation band).

Now consider the restrictions of $\tilde{f}(x)$ to all $(p+2)$-generation bands in D_L and D_R, and use the translation properties for G and its derivative g. Then by applying Lemma 2 with $\|g\| = 2^{-2p+2}$, $d > 2^{-2p+1-(p+2)}$, $m(I) = 2^{-n}$ we obtain

$$\text{(23)}\quad \mu(D_L \cap Q) + \mu(D_R \cap Q) \leqslant \left(1 + \operatorname{int} \frac{2^{-n}}{2^{-2p+1-(p+2)}}\right) \cdot 2^{-2p+2}.$$

The number of bands from the $(p+1)$ generation contained in D_0 are $2^{-n}/2^{-2p+1}$, and, since $2^p < n$ by (18), we have, for $\alpha < 2$,

$$\begin{aligned} \mu(Q) = \mu(B_0 \cap Q) &\leqslant \frac{2^{-n}}{2^{-2p+1}} \cdot \left(1 + \operatorname{int} \frac{2^{-n}}{2^{-2p+1-(p+2)}}\right) \cdot 2^{-2p+2} \\ \text{(24)}\qquad &\leqslant 2^{-n} \cdot 2^{-2p+1} + 2^{-2n+p+2} \leqslant (2^{-n})^2 \cdot (1 + 2^{p+2}) \\ &\leqslant (2^{-n})^2 (1 + 4n) \leqslant (2^{-n})^\alpha = |Q|^\alpha \end{aligned}$$

if $1 + 4n \leqslant 2^{n(2-\alpha)}$.

The Mass Distribution Principle now gives (17) and the proof is complete.

Remark. The nowhere-differentiability of the constructed function f is omitted in the statement of the Theorem. However this property can be established by minor changes to the proof in [RHA] or the proof of Theorem 2-9 in [D-W]. The continuity of $f(x)$ follows from uniform convergence of the series (4).

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