

Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	41 (1995)
Heft:	1-2: L'ENSEIGNEMENT MATHÉMATIQUE
 Artikel:	CONCERNING A REAL-VALUED CONTINUOUS FUNCTION ON THE INTERVAL WITH GRAPH OF HAUSDORFF DIMENSION 2
Autor:	Wingren, Peter
Kapitel:	4. Mass distribution and Hausdorff dimension
DOI:	https://doi.org/10.5169/seals-61818

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 22.07.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Proof. It suffices to assume that $I = [0, m(I)]$. The general case will then follow by a change of variables. If we use the notation $M = \text{int} \frac{m(I)}{d}$ we get

$$\begin{aligned}
 & \int_I \sum_{n=-\infty}^{+\infty} g(y - nd) dy \leq \sum_{m=0}^M \int_{m \cdot d}^{(m+1)d} \sum_{n=-\infty}^{+\infty} g(y - nd) dy \\
 (7) \quad &= \sum_{m=0}^M \sum_{n=-\infty}^{+\infty} \int_{m \cdot d}^{(m+1)d} g(y - nd) dy = \sum_{m=0}^M \sum_{n=-\infty}^{+\infty} \int_{(m-n)d}^{(m-n+1)d} g(t) dt \\
 &= \sum_{m=0}^M \|g\| = \left(1 + \text{int}\left(\frac{m(I)}{d}\right)\right) \|g\|.
 \end{aligned}$$

3. HAUSDORFF MEASURE, NET MEASURE AND HAUSDORFF DIMENSION

This section presents standard results and definitions; see for example [FAL1].

The α -dimensional Hausdorff measure of a subset A of \mathbf{R}^n is defined by

$$(8) \quad H^\alpha(A) = \liminf_{\delta \rightarrow 0} \sum_{\{U_i\}}^{\infty} |U_i|^\alpha,$$

where $\{U_i\}_1^\infty$ is a covering of A with $|U_i| < \delta$, $i = 1, 2, \dots$, and the infimum is taken over all such coverings. The unique number α_0 such that $\alpha < \alpha_0$ implies $H^\alpha(A) = +\infty$ and $\alpha_0 < \alpha$ implies $H^\alpha(A) = 0$ is by definition the Hausdorff dimension of A .

The net measure $M^\alpha(A)$ of A is defined similarly except that the coverings $\{U_i\}$ consist of closed dyadic cubes. It follows that there exists a constant $c_1 > 0$ such that

$$(9) \quad c_1 M^\alpha(A) \leq H^\alpha(A) \leq M^\alpha(A).$$

Since $M^\alpha(A)$ and $H^\alpha(A)$ must therefore yield identical dimensions for A it will suffice to work with dyadic cubes.

4. MASS DISTRIBUTION AND HAUSDORFF DIMENSION

The following well known (see e.g. [FAL2, p. 232]) mass distribution principle will be used in Section 5.

Mass Distribution Principle. Let μ be a mass distribution on $A \subset \mathbf{R}^n$. If there exist constants $c > 0$ and $\delta > 0$ such that, for all dyadic cubes $Q \subset \mathbf{R}^n$ with $|Q| \leq \delta$,

$$(10) \quad \mu(Q) \leq c \cdot |Q|^\alpha,$$

then

$$(11) \quad \alpha \leq \dim_H(A).$$

Proof. Let $\{Q_i\}_{i=1}^\infty$ be a covering of A with dyadic cubes of diameter not exceeding δ . Then

$$(12) \quad 0 < \mu(A) \leq \mu\left(\bigcup_{i=1}^\infty Q_i\right) \leq \sum_1^\infty \mu(Q_i) \leq c \cdot \sum_1^\infty |Q_i|^\alpha$$

and hence the discontinuity in the $M^\alpha(A)$ -graph from $+\infty$ to 0 occurs at a value not less than α . Thus

$$(13) \quad \alpha \leq \dim_H(A).$$

5. THE MAIN RESULT

The notation used in the following theorem and in its proof can be found in Section 0.

THEOREM. *Let*

$$(14) \quad f(x) = \sum_{p=0}^\infty 2^{-p} \operatorname{dist}(2^{2p}x, \mathbf{Z}), \quad x \in [0, 1].$$

Then for every Borel subset B of $\operatorname{graph}(f)$ with $m(\operatorname{Proj}(B)) > 0$,

$$(15) \quad \dim_H(B) = 2.$$

Proof. Assume that B is a Borel set as above. From $\operatorname{graph}(f) \subset \mathbf{R}^2$ there follows

$$(16) \quad \dim_H(B) \leq 2.$$

It will suffice to prove that

$$(17) \quad \alpha \leq \dim_H(B)$$

for an arbitrary positive $\alpha < 2$. Distribute the unit mass as in Lemma 1. Let Q be a dyadic cube with side length less than $\frac{1}{4}$. Then the side length