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Artikel: CONCERNING A REAL-VALUED CONTINUOUS FUNCTION ON THE INTERVAL WITH GRAPH OF HAUSDORFF DIMENSION 2

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Kapitel: 4. Mass distribution and Hausdorff dimension

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Proof. It suffices to assume that $I = [0, m(I)]$. The general case will then follow by a change of variables. If we use the notation $M = \operatorname{int}\frac{m(I)}{d}$ we get

$$\int_I \sum_{n=-\infty}^{+\infty} g(y-nd)\,dy \leqslant \sum_{m=0}^{M} \int_{m\cdot d}^{(m+1)d} \sum_{n=-\infty}^{+\infty} g(y-nd)\,dy$$

$$= \sum_{m=0}^{M} \sum_{n=-\infty}^{+\infty} \int_{m\cdot d}^{(m+1)d} g(y-nd)\,dy = \sum_{m=0}^{M} \sum_{n=-\infty}^{+\infty} \int_{(m-n)d}^{(m-n+1)d} g(t)\,dt \tag{7}$$

$$= \sum_{m=0}^{M} \| g \| = \left(1 + \operatorname{int}\left(\frac{m(I)}{d}\right)\right) \| g \| .$$

3. Hausdorff measure, net measure and Hausdorff dimension

This section presents standard results and definitions; see for example [FAL1].

The α-dimensional Hausdorff measure of a subset A of $\mathbf{R}^n$ is defined by

$$H^\alpha(A) = \lim_{\delta \to 0} \inf_{\{U_i\}} \sum_{i=1}^{\infty} | U_i |^\alpha , \tag{8}$$

where $\{U_i\}_1^\infty$ is a covering of A with $| U_i | < \delta$, $i = 1, 2, \ldots$, and the infimum is taken over all such coverings. The unique number α_0 such that $\alpha < \alpha_0$ implies $H^\alpha(A) = +\infty$ and $\alpha_0 < \alpha$ implies $H^\alpha(A) = 0$ is by definition the Hausdorff dimension of A.

The net measure $M^\alpha(A)$ of A is defined similarly except that the coverings $\{U_i\}$ consist of closed dyadic cubes. It follows that there exists a constant $c_1 > 0$ such that

$$c_1 M^\alpha(A) \leqslant H^\alpha(A) \leqslant M^\alpha(A) . \tag{9}$$

Since $M^\alpha(A)$ and $H^\alpha(A)$ must therefore yield identical dimensions for A it will suffice to work with dyadic cubes.

4. Mass distribution and Hausdorff dimension

The following well known (see e.g. [FAL2, p. 232]) mass distribution principle will be used in Section 5.

Mass Distribution Principle. Let μ *be a mass distribution on* $A \subset \mathbf{R}^n$. *If there exist constants* $c > 0$ *and* $\delta > 0$ *such that, for all dyadic cubes* $Q \subset \mathbf{R}^n$ *with* $|Q| \leqslant \delta$,

$$\mu(Q) \leqslant c \cdot |Q|^{\alpha}, \tag{10}$$

then

$$\alpha \leqslant \dim_H(A). \tag{11}$$

Proof. Let $\{Q_i\}_{i=1}^{\infty}$ be a covering of A with dyadic cubes of diameter not exceeding δ. Then

$$0 < \mu(A) \leqslant \mu\left(\bigcup_{i=1}^{\infty} Q_i\right) \leqslant \sum_{1}^{\infty} \mu(Q_i) \leqslant c \cdot \sum_{1}^{\infty} |Q_i|^{\alpha} \tag{12}$$

and hence the discontinuity in the $M^{\alpha}(A)$-graph from $+\infty$ to 0 occurs at a value not less than α. Thus

$$\alpha \leqslant \dim_H(A). \tag{13}$$

5. THE MAIN RESULT

The notation used in the following theorem and in its proof can be found in Section 0.

THEOREM. *Let*

$$f(x) = \sum_{p=0}^{\infty} 2^{-p} \operatorname{dist}(2^{2^p} x, \mathbf{Z}), \quad x \in [0, 1]. \tag{14}$$

Then for every Borel subset B *of* graph(f) *with* $m(\operatorname{Proj}(B)) > 0$,

$$\dim_H(B) = 2. \tag{15}$$

Proof. Assume that B is a Borel set as above. From graph(f) $\subset \mathbf{R}^2$ there follows

$$\dim_H(B) \leqslant 2. \tag{16}$$

It will suffice to prove that

$$\alpha \leqslant \dim_H(B) \tag{17}$$

for an arbitrary positive $\alpha < 2$. Distribute the unit mass as in Lemma 1. Let Q be a dyadic cube with side length less than $\frac{1}{4}$. Then the side length