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Proof. It suffices to assume that $I = [0, m(I)]$. The general case will then follow by a change of variables. If we use the notation $M = \text{int} \frac{m(I)}{d}$ we get

$$\begin{aligned}
 \int_I \sum_{n=-\infty}^{+\infty} g(y - nd) dy &\leq \sum_{m=0}^M \int_{m \cdot d}^{(m+1)d} \sum_{n=-\infty}^{+\infty} g(y - nd) dy \\
 (7) \quad &= \sum_{m=0}^M \sum_{n=-\infty}^{+\infty} \int_{m \cdot d}^{(m+1)d} g(y - nd) dy = \sum_{m=0}^M \sum_{n=-\infty}^{+\infty} \int_{(m-n)d}^{(m-n+1)d} g(t) dt \\
 &= \sum_{m=0}^M \|g\| = \left(1 + \text{int} \left(\frac{m(I)}{d}\right)\right) \|g\|.
 \end{aligned}$$

3. HAUSDORFF MEASURE, NET MEASURE AND HAUSDORFF DIMENSION

This section presents standard results and definitions; see for example [FAL1].

The α -dimensional Hausdorff measure of a subset A of \mathbf{R}^n is defined by

$$(8) \quad H^\alpha(A) = \lim_{\delta \rightarrow 0} \inf_{\{U_i\}} \sum_{i=1}^{\infty} |U_i|^\alpha,$$

where $\{U_i\}_1^\infty$ is a covering of A with $|U_i| < \delta$, $i = 1, 2, \dots$, and the infimum is taken over all such coverings. The unique number α_0 such that $\alpha < \alpha_0$ implies $H^\alpha(A) = +\infty$ and $\alpha_0 < \alpha$ implies $H^\alpha(A) = 0$ is by definition the Hausdorff dimension of A .

The net measure $M^\alpha(A)$ of A is defined similarly except that the coverings $\{U_i\}$ consist of closed dyadic cubes. It follows that there exists a constant $c_1 > 0$ such that

$$(9) \quad c_1 M^\alpha(A) \leq H^\alpha(A) \leq M^\alpha(A).$$

Since $M^\alpha(A)$ and $H^\alpha(A)$ must therefore yield identical dimensions for A it will suffice to work with dyadic cubes.

4. MASS DISTRIBUTION AND HAUSDORFF DIMENSION

The following well known (see e.g. [FAL2, p. 232]) mass distribution principle will be used in Section 5.

Mass Distribution Principle. Let μ be a mass distribution on $A \subset \mathbf{R}^n$. If there exist constants $c > 0$ and $\delta > 0$ such that, for all dyadic cubes $Q \subset \mathbf{R}^n$ with $|Q| \leq \delta$,

$$(10) \quad \mu(Q) \leq c \cdot |Q|^\alpha,$$

then

$$(11) \quad \alpha \leq \dim_H(A).$$

Proof. Let $\{Q_i\}_{i=1}^\infty$ be a covering of A with dyadic cubes of diameter not exceeding δ . Then

$$(12) \quad 0 < \mu(A) \leq \mu\left(\bigcup_{i=1}^\infty Q_i\right) \leq \sum_1^\infty \mu(Q_i) \leq c \cdot \sum_1^\infty |Q_i|^\alpha$$

and hence the discontinuity in the $M^\alpha(A)$ -graph from $+\infty$ to 0 occurs at a value not less than α . Thus

$$(13) \quad \alpha \leq \dim_H(A).$$

5. THE MAIN RESULT

The notation used in the following theorem and in its proof can be found in Section 0.

THEOREM. *Let*

$$(14) \quad f(x) = \sum_{p=0}^\infty 2^{-p} \text{dist}(2^{2^p}x, \mathbf{Z}), \quad x \in [0, 1].$$

Then for every Borel subset B of $\text{graph}(f)$ with $m(\text{Proj}(B)) > 0$,

$$(15) \quad \dim_H(B) = 2.$$

Proof. Assume that B is a Borel set as above. From $\text{graph}(f) \subset \mathbf{R}^2$ there follows

$$(16) \quad \dim_H(B) \leq 2.$$

It will suffice to prove that

$$(17) \quad \alpha \leq \dim_H(B)$$

for an arbitrary positive $\alpha < 2$. Distribute the unit mass as in Lemma 1. Let Q be a dyadic cube with side length less than $\frac{1}{4}$. Then the side length