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Artikel: CONCERNING A REAL-VALUED CONTINUOUS FUNCTION ON THE INTERVAL WITH GRAPH OF HAUSDORFF DIMENSION 2
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Kapitel: 3. Hausdorff measure, net measure and Hausdorff dimension
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Proof. It suffices to assume that $I = [0, m(I)]$. The general case will then follow by a change of variables. If we use the notation $M = \operatorname{int}\frac{m(I)}{d}$ we get

$$\int_I \sum_{n=-\infty}^{+\infty} g(y-nd)dy \leqslant \sum_{m=0}^{M} \int_{m\cdot d}^{(m+1)d} \sum_{n=-\infty}^{+\infty} g(y-nd)dy$$

$$= \sum_{m=0}^{M} \sum_{n=-\infty}^{+\infty} \int_{m\cdot d}^{(m+1)d} g(y-nd)dy = \sum_{m=0}^{M} \sum_{n=-\infty}^{+\infty} \int_{(m-n)d}^{(m-n+1)d} g(t)dt \tag{7}$$

$$= \sum_{m=0}^{M} \|g\| = \left(1 + \operatorname{int}\left(\frac{m(I)}{d}\right)\right)\|g\| .$$

3. Hausdorff measure, net measure and Hausdorff dimension

This section presents standard results and definitions; see for example [FAL1].

The α-dimensional Hausdorff measure of a subset A of $\mathbf{R}^n$ is defined by

$$H^\alpha(A) = \lim_{\delta\to 0} \inf_{\{U_i\}} \sum_{i=1}^{\infty} |U_i|^\alpha , \tag{8}$$

where $\{U_i\}_1^\infty$ is a covering of A with $|U_i| < \delta$, $i = 1, 2, \ldots$, and the infimum is taken over all such coverings. The unique number α_0 such that $\alpha < \alpha_0$ implies $H^\alpha(A) = +\infty$ and $\alpha_0 < \alpha$ implies $H^\alpha(A) = 0$ is by definition the Hausdorff dimension of A.

The net measure $M^\alpha(A)$ of A is defined similarly except that the coverings $\{U_i\}$ consist of closed dyadic cubes. It follows that there exists a constant $c_1 > 0$ such that

$$c_1 M^\alpha(A) \leqslant H^\alpha(A) \leqslant M^\alpha(A) . \tag{9}$$

Since $M^\alpha(A)$ and $H^\alpha(A)$ must therefore yield identical dimensions for A it will suffice to work with dyadic cubes.

4. Mass distribution and Hausdorff dimension

The following well known (see e.g. [FAL2, p. 232]) mass distribution principle will be used in Section 5.