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constructed by closed dyadic cubes. The graph of a real valued function $f \in C[0, 1]$ is denoted by $\text{graph}(f)$. By a dyadic cube we mean a cube which is the Cartesian product of dyadic intervals. If Q is an arbitrary dyadic closed cube, then the band of type $\{(x, y) : (x, z) \in Q \text{ for some } z \in \mathbf{R}\}$ is called a dyadic band. In our construction the dyadic bands of width 2^{-2^p} play a special role. They are called bands of generation $p, p = 0, 1, 2, \dots$.

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1. A LEMMA ABOUT MASS DISTRIBUTION

By a mass distribution on a subset A of \mathbf{R}^2 we mean a measure μ on A such that $0 < \mu(A) < \infty$.

LEMMA 1. *Let f be a real valued measurable function defined on $[0, 1]$. Then there is a mass distribution μ on $F := \text{graph}(f)$ such that*

1) *for any two subintervals I and I' of $[0, 1]$, with $m(I) = m(I')$,*

$$\mu(I \times \mathbf{R}) = \mu(I' \times \mathbf{R})$$

and

2) *if for two Borel sets B_1 and B_2 in $[0, 1] \times \mathbf{R}$ there exists $(x_0, y_0) \in \mathbf{R}^2$ such that*

$$B_1 \cap F + (x_0, y_0) = B_2 \cap F$$

then

$$\mu(B_1) = \mu(B_2).$$

Proof. Let B be an arbitrary Borel set in \mathbf{R}^2 . Define

$$(5) \quad \mu(B) = m(\tilde{f}^{-1}(B)).$$

Then it is obvious that μ is a mass distribution on $\text{graph}(f)$ and 1) and 2) follow from the translation invariance of the Lebesgue measure.

2. A LEMMA ABOUT MASS DISTRIBUTION AND SUCCESSIVE TRANSLATIONS

LEMMA 2. *Let $g(y) \geq 0$ and $g(y) \in L^1(\mathbf{R})$. If I is a finite interval and d is a positive real number then*

$$(6) \quad \int_I \sum_{n=-\infty}^{\infty} g(y - nd) dy < \left(1 + \text{int} \frac{m(I)}{d}\right) \cdot \|g\|.$$

Proof. It suffices to assume that $I = [0, m(I)]$. The general case will then follow by a change of variables. If we use the notation $M = \text{int} \frac{m(I)}{d}$ we get

$$\begin{aligned}
 \int_I \sum_{n=-\infty}^{+\infty} g(y - nd) dy &\leq \sum_{m=0}^M \int_{m \cdot d}^{(m+1)d} \sum_{n=-\infty}^{+\infty} g(y - nd) dy \\
 (7) \quad &= \sum_{m=0}^M \sum_{n=-\infty}^{+\infty} \int_{m \cdot d}^{(m+1)d} g(y - nd) dy = \sum_{m=0}^M \sum_{n=-\infty}^{+\infty} \int_{(m-n)d}^{(m-n+1)d} g(t) dt \\
 &= \sum_{m=0}^M \|g\| = \left(1 + \text{int} \left(\frac{m(I)}{d}\right)\right) \|g\|.
 \end{aligned}$$

3. HAUSDORFF MEASURE, NET MEASURE AND HAUSDORFF DIMENSION

This section presents standard results and definitions; see for example [FAL1].

The α -dimensional Hausdorff measure of a subset A of \mathbf{R}^n is defined by

$$(8) \quad H^\alpha(A) = \lim_{\delta \rightarrow 0} \inf_{\{U_i\}} \sum_{i=1}^{\infty} |U_i|^\alpha,$$

where $\{U_i\}_1^\infty$ is a covering of A with $|U_i| < \delta$, $i = 1, 2, \dots$, and the infimum is taken over all such coverings. The unique number α_0 such that $\alpha < \alpha_0$ implies $H^\alpha(A) = +\infty$ and $\alpha_0 < \alpha$ implies $H^\alpha(A) = 0$ is by definition the Hausdorff dimension of A .

The net measure $M^\alpha(A)$ of A is defined similarly except that the coverings $\{U_i\}$ consist of closed dyadic cubes. It follows that there exists a constant $c_1 > 0$ such that

$$(9) \quad c_1 M^\alpha(A) \leq H^\alpha(A) \leq M^\alpha(A).$$

Since $M^\alpha(A)$ and $H^\alpha(A)$ must therefore yield identical dimensions for A it will suffice to work with dyadic cubes.

4. MASS DISTRIBUTION AND HAUSDORFF DIMENSION

The following well known (see e.g. [FAL2, p. 232]) mass distribution principle will be used in Section 5.