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## CONCERNING A REAL-VALUED CONTINUOUS FUNCTION ON THE INTERVAL WITH GRAPH OF HAUSDORFF DIMENSION 2

by Peter WINGREN

**ABSTRACT.** A real-valued continuous nowhere-differentiable function on  $[0, 1]$  is constructed. Its graph  $F$  is proved to have the following property. If  $B$  is a Borel subset of  $F$  and if the projection of  $B$  on  $[0, 1]$  has positive Lebesgue measure, then the Hausdorff dimension of  $B$  is two.

### 0. INTRODUCTION

In 1903 Takagi [TAK, p. 176] gave an extremely simple construction of a nowhere differentiable real-valued continuous function on  $[0, 1]$ . Takagi's construction is

$$(1) \quad T(x) = \sum_{p=0}^{\infty} 2^{-p} \operatorname{dist}(2^p x, \mathbf{Z})$$

where each term is a scaled version of the sawtooth function

$$(2) \quad \operatorname{dist}(x, \mathbf{Z}) := \inf \{ |x - y| : y \in \mathbf{Z} \}.$$

Later, in 1930, van der Waerden [WAE] gave a similar example, which de Rham [RHA], in 1957, improved to an example identical with Takagi's.

It follows from a proof of Mauldin and Williams [M-W, pp. 795-797] that the graph of the Takagi function has a  $\sigma$ -finite linear Hausdorff measure and hence is of Hausdorff dimension 1.

In 1937 Besicovitch and Ursell [B-U, p. 29] constructed for an arbitrary  $\alpha$ ,  $1 < \alpha < 2$ , a real-valued nowhere-differentiable function in  $C[0, 1]$  with graph of Hausdorff dimension  $\alpha$ . They too used the sawtooth function  $\operatorname{dist}(x, \mathbf{Z})$  as a building block in their construction.

In this paper we construct a real valued continuous function  $f(x)$ ,  $x \in [0, 1]$ , whose graph has an optimal property with respect to Hausdorff dimension and measure.

We prove that for an arbitrary  $\alpha$ ,  $1 < \alpha < 2$ ,  $f(x)$  has the property

$\mathcal{P}(\alpha)$ : Every Borel subset  $B \subset \text{graph}(f)$ , with projection on the  $x$ -axis of positive Lebesgue measure  $m(\text{Proj}(B)) > 0$ , has infinite  $\alpha$ -dimensional Hausdorff measure

$$(3) \quad H^\alpha(B) = +\infty.$$

It is easy to see that

$$\mathcal{P}(\alpha) \forall \alpha < 2 \Leftrightarrow \mathcal{P}$$

where

$\mathcal{P}$ : Every Borel set  $B \subset \text{graph}(f)$  with  $m(\text{Proj}(B)) > 0$  has Hausdorff dimension equal to two.

Rather than establish a general theorem valid for a class of functions we shall construct a single function with the desired property. The rationale is to provide a simple construction accompanied by a short, clear and instructive proof.

Our function is

$$(4) \quad f(x) = \sum_{p=0}^{\infty} 2^{-p} \text{dist}(2^{2^p} x, \mathbf{Z}).$$

Even though  $\mathcal{P}$  is established for only a single function  $f$ , the proof contains general methods extracted as Lemma 1 and Lemma 2. It appears that Lemma 1 is well known in more general cases than ours; compare [P-U, p. 159, the beginning of the proof of their Lemma 1]. However the proof is included here for completeness and because in the present case it is particularly simple.

The author is grateful to Professor V.P. Havin [HAV] for suggesting the investigation of fractal graphs with respect to  $\mathcal{P}(\alpha)$ ,  $\alpha = 1$ .

**PROBLEM.** We believe that the following problem is unsolved.

*Part 1:* Construct a real valued function in  $C[0, 1]$  with graph of Hausdorff dimension 1 and with property  $\mathcal{P}(\alpha)$  for  $\alpha = 1$ .

*Part 2:* Determine the optimal smoothness in terms of the second difference of such a function.

*Notation.* The diameter of  $U$  is denoted by  $|U|$  and the  $L^1$ -norm of  $g \in L^1(\mathbf{R})$  by  $\|g\|$ . If  $f$  is a real valued function in  $C[0, 1]$ , we write  $\tilde{f}(x)$  for  $(x, f(x))$ . The notation  $H^\alpha(F)$  stands for  $\alpha$ -dimensional Hausdorff measure of a set  $F \subset \mathbf{R}^2$  and  $M^\alpha(F)$  is the  $\alpha$ -dimensional net measure of  $F$

constructed by closed dyadic cubes. The graph of a real valued function  $f \in C[0, 1]$  is denoted by  $\text{graph}(f)$ . By a dyadic cube we mean a cube which is the Cartesian product of dyadic intervals. If  $Q$  is an arbitrary dyadic closed cube, then the band of type  $\{(x, y) : (x, z) \in Q \text{ for some } z \in \mathbf{R}\}$  is called a dyadic band. In our construction the dyadic bands of width  $2^{-2^p}$  play a special role. They are called bands of generation  $p, p = 0, 1, 2, \dots$ .

*Acknowledgement.* We would like to thank the referee for helpful suggestions.

## 1. A LEMMA ABOUT MASS DISTRIBUTION

By a mass distribution on a subset  $A$  of  $\mathbf{R}^2$  we mean a measure  $\mu$  on  $A$  such that  $0 < \mu(A) < \infty$ .

LEMMA 1. *Let  $f$  be a real valued measurable function defined on  $[0, 1]$ . Then there is a mass distribution  $\mu$  on  $F := \text{graph}(f)$  such that*

1) *for any two subintervals  $I$  and  $I'$  of  $[0, 1]$ , with  $m(I) = m(I')$ ,*

$$\mu(I \times \mathbf{R}) = \mu(I' \times \mathbf{R})$$

*and*

2) *if for two Borel sets  $B_1$  and  $B_2$  in  $[0, 1] \times \mathbf{R}$  there exists  $(x_0, y_0) \in \mathbf{R}^2$  such that*

$$B_1 \cap F + (x_0, y_0) = B_2 \cap F$$

*then*

$$\mu(B_1) = \mu(B_2).$$

*Proof.* Let  $B$  be an arbitrary Borel set in  $\mathbf{R}^2$ . Define

$$(5) \quad \mu(B) = m(\tilde{f}^{-1}(B)).$$

Then it is obvious that  $\mu$  is a mass distribution on  $\text{graph}(f)$  and 1) and 2) follow from the translation invariance of the Lebesgue measure.

## 2. A LEMMA ABOUT MASS DISTRIBUTION AND SUCCESSIVE TRANSLATIONS

LEMMA 2. *Let  $g(y) \geq 0$  and  $g(y) \in L^1(\mathbf{R})$ . If  $I$  is a finite interval and  $d$  is a positive real number then*

$$(6) \quad \int_I \sum_{n=-\infty}^{\infty} g(y - nd) dy < \left(1 + \text{int} \frac{m(I)}{d}\right) \cdot \|g\|.$$