

<b>Zeitschrift:</b>	L'Enseignement Mathématique
<b>Herausgeber:</b>	Commission Internationale de l'Enseignement Mathématique
<b>Band:</b>	41 (1995)
<b>Heft:</b>	1-2: L'ENSEIGNEMENT MATHÉMATIQUE
 <b>Artikel:</b>	CONCERNING A REAL-VALUED CONTINUOUS FUNCTION ON THE INTERVAL WITH GRAPH OF HAUSDORFF DIMENSION 2
<b>Autor:</b>	Wingren, Peter
<b>DOI:</b>	<a href="https://doi.org/10.5169/seals-61818">https://doi.org/10.5169/seals-61818</a>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 24.01.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## CONCERNING A REAL-VALUED CONTINUOUS FUNCTION ON THE INTERVAL WITH GRAPH OF HAUSDORFF DIMENSION 2

by Peter WINGREN

ABSTRACT. A real-valued continuous nowhere-differentiable function on  $[0, 1]$  is constructed. Its graph  $F$  is proved to have the following property. If  $B$  is a Borel subset of  $F$  and if the projection of  $B$  on  $[0, 1]$  has positive Lebesgue measure, then the Hausdorff dimension of  $B$  is two.

### 0. INTRODUCTION

In 1903 Takagi [TAK, p. 176] gave an extremely simple construction of a nowhere differentiable real-valued continuous function on  $[0, 1]$ . Takagi's construction is

$$(1) \quad T(x) = \sum_{p=0}^{\infty} 2^{-p} \text{dist}(2^p x, \mathbf{Z})$$

where each term is a scaled version of the sawtooth function

$$(2) \quad \text{dist}(x, \mathbf{Z}) := \inf \{ |x - y| : y \in \mathbf{Z} \} .$$

Later, in 1930, van der Waerden [WAE] gave a similar example, which de Rham [RHA], in 1957, improved to an example identical with Takagi's.

It follows from a proof of Mauldin and Williams [M-W, pp. 795-797] that the graph of the Takagi function has a  $\sigma$ -finite linear Hausdorff measure and hence is of Hausdorff dimension 1.

In 1937 Besicovitch and Ursell [B-U, p. 29] constructed for an arbitrary  $\alpha$ ,  $1 < \alpha < 2$ , a real-valued nowhere-differentiable function in  $C[0, 1]$  with graph of Hausdorff dimension  $\alpha$ . They too used the sawtooth function  $\text{dist}(x, \mathbf{Z})$  as a building block in their construction.

In this paper we construct a real valued continuous function  $f(x)$ ,  $x \in [0, 1]$ , whose graph has an optimal property with respect to Hausdorff dimension and measure.

We prove that for an arbitrary  $\alpha$ ,  $1 < \alpha < 2$ ,  $f(x)$  has the property

$\mathcal{P}(\alpha)$ : Every Borel subset  $B \subset \text{graph}(f)$ , with projection on the  $x$ -axis of positive Lebesgue measure  $m(\text{Proj}(B)) > 0$ , has infinite  $\alpha$ -dimensional Hausdorff measure

$$(3) \quad H^\alpha(B) = +\infty.$$

It is easy to see that

$$\mathcal{P}(\alpha) \forall \alpha < 2 \Leftrightarrow \mathcal{P}$$

where

$\mathcal{P}$ : Every Borel set  $B \subset \text{graph}(f)$  with  $m(\text{Proj}(B)) > 0$  has Hausdorff dimension equal to two.

Rather than establish a general theorem valid for a class of functions we shall construct a single function with the desired property. The rationale is to provide a simple construction accompanied by a short, clear and instructive proof.

Our function is

$$(4) \quad f(x) = \sum_{p=0}^{\infty} 2^{-p} \text{dist}(2^{2p}x, \mathbf{Z}).$$

Even though  $\mathcal{P}$  is established for only a single function  $f$ , the proof contains general methods extracted as Lemma 1 and Lemma 2. It appears that Lemma 1 is well known in more general cases than ours; compare [P-U, p. 159, the beginning of the proof of their Lemma 1]. However the proof is included here for completeness and because in the present case it is particularly simple.

The author is grateful to Professor V.P. Havin [HAV] for suggesting the investigation of fractal graphs with respect to  $\mathcal{P}(\alpha)$ ,  $\alpha = 1$ .

**PROBLEM.** We believe that the following problem is unsolved.

*Part 1:* Construct a real valued function in  $C[0, 1]$  with graph of Hausdorff dimension 1 and with property  $\mathcal{P}(\alpha)$  for  $\alpha = 1$ .

*Part 2:* Determine the optimal smoothness in terms of the second difference of such a function.

*Notation.* The diameter of  $U$  is denoted by  $|U|$  and the  $L^1$ -norm of  $g \in L^1(\mathbf{R})$  by  $\|g\|$ . If  $f$  is a real valued function in  $C[0, 1]$ , we write  $\tilde{f}(x)$  for  $(x, f(x))$ . The notation  $H^\alpha(F)$  stands for  $\alpha$ -dimensional Hausdorff measure of a set  $F \subset \mathbf{R}^2$  and  $M^\alpha(F)$  is the  $\alpha$ -dimensional net measure of  $F$ .

constructed by closed dyadic cubes. The graph of a real valued function  $f \in C[0, 1]$  is denoted by  $\text{graph}(f)$ . By a dyadic cube we mean a cube which is the Cartesian product of dyadic intervals. If  $Q$  is an arbitrary dyadic closed cube, then the band of type  $\{(x, y) : (x, z) \in Q \text{ for some } z \in \mathbf{R}\}$  is called a dyadic band. In our construction the dyadic bands of width  $2^{-2^p}$  play a special role. They are called bands of generation  $p$ ,  $p = 0, 1, 2, \dots$ .

*Acknowledgement.* We would like to thank the referee for helpful suggestions.

### 1. A LEMMA ABOUT MASS DISTRIBUTION

By a mass distribution on a subset  $A$  of  $\mathbf{R}^2$  we mean a measure  $\mu$  on  $A$  such that  $0 < \mu(A) < \infty$ .

LEMMA 1. *Let  $f$  be a real valued measurable function defined on  $[0, 1]$ . Then there is a mass distribution  $\mu$  on  $F := \text{graph}(f)$  such that*

1) *for any two subintervals  $I$  and  $I'$  of  $[0, 1]$ , with  $m(I) = m(I')$ ,*

$$\mu(I \times \mathbf{R}) = \mu(I' \times \mathbf{R})$$

*and*

2) *if for two Borel sets  $B_1$  and  $B_2$  in  $[0, 1] \times \mathbf{R}$  there exists  $(x_0, y_0) \in \mathbf{R}^2$  such that*

$$B_1 \cap F + (x_0, y_0) = B_2 \cap F$$

*then*

$$\mu(B_1) = \mu(B_2).$$

*Proof.* Let  $B$  be an arbitrary Borel set in  $\mathbf{R}^2$ . Define

$$(5) \quad \mu(B) = m(\tilde{f}^{-1}(B)).$$

Then it is obvious that  $\mu$  is a mass distribution on  $\text{graph}(f)$  and 1) and 2) follow from the translation invariance of the Lebesgue measure.

### 2. A LEMMA ABOUT MASS DISTRIBUTION AND SUCCESSIVE TRANSLATIONS

LEMMA 2. *Let  $g(y) \geq 0$  and  $g(y) \in L^1(\mathbf{R})$ . If  $I$  is a finite interval and  $d$  is a positive real number then*

$$(6) \quad \int_I \sum_{n=-\infty}^{\infty} g(y - nd) dy < \left(1 + \text{int} \frac{m(I)}{d}\right) \cdot \|g\|.$$

*Proof.* It suffices to assume that  $I = [0, m(I)]$ . The general case will then follow by a change of variables. If we use the notation  $M = \text{int} \frac{m(I)}{d}$  we get

$$\begin{aligned}
 \int_I \sum_{n=-\infty}^{+\infty} g(y - nd) dy &\leq \sum_{m=0}^M \int_{m \cdot d}^{(m+1)d} \sum_{n=-\infty}^{+\infty} g(y - nd) dy \\
 (7) \quad &= \sum_{m=0}^M \sum_{n=-\infty}^{+\infty} \int_{m \cdot d}^{(m+1)d} g(y - nd) dy = \sum_{m=0}^M \sum_{n=-\infty}^{+\infty} \int_{(m-n)d}^{(m-n+1)d} g(t) dt \\
 &= \sum_{m=0}^M \|g\| = \left(1 + \text{int}\left(\frac{m(I)}{d}\right)\right) \|g\|.
 \end{aligned}$$

### 3. HAUSDORFF MEASURE, NET MEASURE AND HAUSDORFF DIMENSION

This section presents standard results and definitions; see for example [FAL1].

The  $\alpha$ -dimensional Hausdorff measure of a subset  $A$  of  $\mathbf{R}^n$  is defined by

$$(8) \quad H^\alpha(A) = \lim_{\delta \rightarrow 0} \inf_{\{U_i\}} \sum_{i=1}^{\infty} |U_i|^\alpha,$$

where  $\{U_i\}_1^\infty$  is a covering of  $A$  with  $|U_i| < \delta$ ,  $i = 1, 2, \dots$ , and the infimum is taken over all such coverings. The unique number  $\alpha_0$  such that  $\alpha < \alpha_0$  implies  $H^\alpha(A) = +\infty$  and  $\alpha_0 < \alpha$  implies  $H^\alpha(A) = 0$  is by definition the Hausdorff dimension of  $A$ .

The net measure  $M^\alpha(A)$  of  $A$  is defined similarly except that the coverings  $\{U_i\}$  consist of closed dyadic cubes. It follows that there exists a constant  $c_1 > 0$  such that

$$(9) \quad c_1 M^\alpha(A) \leq H^\alpha(A) \leq M^\alpha(A).$$

Since  $M^\alpha(A)$  and  $H^\alpha(A)$  must therefore yield identical dimensions for  $A$  it will suffice to work with dyadic cubes.

### 4. MASS DISTRIBUTION AND HAUSDORFF DIMENSION

The following well known (see e.g. [FAL2, p. 232]) mass distribution principle will be used in Section 5.

*Mass Distribution Principle.* Let  $\mu$  be a mass distribution on  $A \subset \mathbf{R}^n$ . If there exist constants  $c > 0$  and  $\delta > 0$  such that, for all dyadic cubes  $Q \subset \mathbf{R}^n$  with  $|Q| \leq \delta$ ,

$$(10) \quad \mu(Q) \leq c \cdot |Q|^\alpha,$$

then

$$(11) \quad \alpha \leq \dim_H(A).$$

*Proof.* Let  $\{Q_i\}_{i=1}^\infty$  be a covering of  $A$  with dyadic cubes of diameter not exceeding  $\delta$ . Then

$$(12) \quad 0 < \mu(A) \leq \mu\left(\bigcup_{i=1}^\infty Q_i\right) \leq \sum_1^\infty \mu(Q_i) \leq c \cdot \sum_1^\infty |Q_i|^\alpha$$

and hence the discontinuity in the  $M^\alpha(A)$ -graph from  $+\infty$  to 0 occurs at a value not less than  $\alpha$ . Thus

$$(13) \quad \alpha \leq \dim_H(A).$$

## 5. THE MAIN RESULT

The notation used in the following theorem and in its proof can be found in Section 0.

**THEOREM.** *Let*

$$(14) \quad f(x) = \sum_{p=0}^\infty 2^{-p} \operatorname{dist}(2^{2p}x, \mathbf{Z}), \quad x \in [0, 1].$$

*Then for every Borel subset  $B$  of  $\operatorname{graph}(f)$  with  $m(\operatorname{Proj}(B)) > 0$ ,*

$$(15) \quad \dim_H(B) = 2.$$

*Proof.* Assume that  $B$  is a Borel set as above. From  $\operatorname{graph}(f) \subset \mathbf{R}^2$  there follows

$$(16) \quad \dim_H(B) \leq 2.$$

It will suffice to prove that

$$(17) \quad \alpha \leq \dim_H(B)$$

for an arbitrary positive  $\alpha < 2$ . Distribute the unit mass as in Lemma 1. Let  $Q$  be a dyadic cube with side length less than  $\frac{1}{4}$ . Then the side length

of  $Q$  is  $2^{-n}$  for some positive integer  $n$  and there is a positive integer  $p$  such that

$$(18) \quad 2^{-2p+1} \leq 2^{-n} < 2^{-2p}.$$

Let  $D_0$  be the smallest vertical band which inscribes  $Q$ , and so it has band width  $2^{-n}$ . From the second inequality in (18) we conclude that  $D_0$  is contained in a band from generation  $p$ . In the discussion and in the estimations which follow, just those bands which are of generations  $p, p+1$  and  $p+2$  play a role. We let  $D$  and  $D_L$  denote an arbitrary band from generation  $p+1$  and its left half, respectively. On  $D_L$  we study  $f(x)$  as a sum of two terms,

$$(19) \quad f(x) = \sum_0^{p+1} 2^{-k} \operatorname{dist}(2^{2k}x, \mathbf{Z}) + \sum_{p+2}^{\infty} 2^{-k} \operatorname{dist}(2^{2k}x, \mathbf{Z}).$$

The first term is linear and the second periodic (one cycle on each subband from generation  $p+2$ ). This implies that the distribution of mass via (5) on each  $(p+2)$ -subband (of  $D_L$ ) is the same but translated a fixed distance  $d_D$  in  $y$ -direction. Now let  $D'$  be a  $(p+2)$ -subband of  $D_L$  and define

$$(20) \quad G_{D'}(y) := \mu(\{(x_1, x_2) \in D' \text{ and } x_2 \leq y\}).$$

Then its derivative  $g(y)$  exists a.e. and

$$(21) \quad \|g\| = 2^{-2p+2}.$$

If  $D'$  and  $D''$  are neighbouring  $(p+2)$ -generation subbands of  $D_L$ , then  $G_{D''}(y)$  is a translation of  $G_{D'}(y)$  by  $d_D$ . Hence, we may use just one function  $G$  and its translates. The fixed translation  $d_D$  of mass in  $y$ -direction from one band to the next may be estimated by the derivative of the first sum of (19),

$$(22) \quad \begin{aligned} d_D &= 2^{-2p+2} \times \left| \frac{d}{dx} \left( \sum_{p=0}^{p+1} 2^{-k} \operatorname{dist}(2^{2k}x, \mathbf{Z}) \right) \right| \\ &\geq 2^{-2p+2} (2^{-(p+1)+2p+1} - 2^{-p+2p} - \dots - 2) > 2^{-2p+1-(p+2)}. \end{aligned}$$

The last inequality holds for  $p > 1$ , because the rapid decrease of the successive terms in the parenthesis implies that its value is larger than half the first term. (It is easy to check that this estimation also works for  $(p+2)$ -generation bands in the right band half  $D_R$  of a  $(p+1)$ -generation band).

Now consider the restrictions of  $\tilde{f}(x)$  to all  $(p+2)$ -generation bands in  $D_L$  and  $D_R$ , and use the translation properties for  $G$  and its derivative  $g$ . Then by applying Lemma 2 with  $\|g\| = 2^{-2p+2}$ ,  $d > 2^{-2p+1-(p+2)}$ ,  $m(I) = 2^{-n}$  we obtain

$$(23) \quad \mu(D_L \cap Q) + \mu(D_R \cap Q) \leq \left(1 + \text{int} \frac{2^{-n}}{2^{-2p+1-(p+2)}}\right) \cdot 2^{-2p+2}.$$

The number of bands from the  $(p+1)$  generation contained in  $D_0$  are  $2^{-n}/2^{-2p+1}$ , and, since  $2^p < n$  by (18), we have, for  $\alpha < 2$ ,

$$\begin{aligned} \mu(Q) = \mu(B_0 \cap Q) &\leq \frac{2^{-n}}{2^{-2p+1}} \cdot \left(1 + \text{int} \frac{2^{-n}}{2^{-2p+1-(p+2)}}\right) \cdot 2^{-2p+2} \\ (24) \quad &\leq 2^{-n} \cdot 2^{-2p+1} + 2^{-2n+p+2} \leq (2^{-n})^2 \cdot (1 + 2^{p+2}) \\ &\leq (2^{-n})^2 (1 + 4n) \leq (2^{-n})^\alpha = |Q|^\alpha \end{aligned}$$

if  $1 + 4n \leq 2^{n(2-\alpha)}$ .

The Mass Distribution Principle now gives (17) and the proof is complete.

*Remark.* The nowhere-differentiability of the constructed function  $f$  is omitted in the statement of the Theorem. However this property can be established by minor changes to the proof in [RHA] or the proof of Theorem 2-9 in [D-W]. The continuity of  $f(x)$  follows from uniform convergence of the series (4).

## REFERENCES

- [B-U] BESICOVICH, A.S. and H.D. URSELL. Sets of fractional dimensions, V: On dimensional numbers of some continuous curves. *Journal of the London Mathematical Society* 12 (1937), 18-25.
- [D-W] DELIU, A. and P. WINGREN. The Takagi operator, Bernoulli sequences, smoothness conditions and fractal curves. *Proc. Amer. Math. Soc.* 121 (1994), 871-881.
- [FAL1] FALCONER, K.J. *The Geometry of Fractal Sets*. Cambridge Tracts in Math. 85, Cambridge Univ. Press, 1985.
- [FAL2] —— Dimensions — their determination and properties. *Fractal Geometry and Analysis*. (Jacques Belair and Serge Dubuc, editors), Kluwer, 1991, 221-254.
- [HAV] HAVIN, V.P. St. Petersburg University, Russia, personal communication.
- [M-W] MAULDIN, R.D. and S.C. WILLIAMS. On the Hausdorff dimension of some graphs. *Trans. Amer. Math. Soc.* 298 (1986), 793-803.
- [P-U] PRZTYCKI, F. and M. URBANSKI. On the Hausdorff Dimension of some fractal sets. *Studia Mathematica XCIII* (1989), 155-186.

- [RHA] de RHAM, G. Sur un exemple de fonction continue sans dérivée. *L'Enseignement Mathématique* 3 (1957), 71-72.
- [TAK] TAKAGI, T. A simple example of a continuous function without derivative. *Proc. Phys. Math. Soc. Japan* 1 (1903), 176-177.
- [WAE] van der WAERDEN, B.L. Ein einfaches Beispiel einer nicht-differenzierbaren stetigen Funktion. *Math. Z.* 32 (1930), 474-475.

(Reçu le 22 février 1994)

Peter Wingren

Umeå University  
Department of Mathematics  
S-901 87 Umeå (Sweden)