

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 41 (1995)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: HIGHER EULER CHARACTERISTICS (I)
Autor: Geoghegan, Ross / Nicas, Andrew
Kapitel: 8. OUTER AUTOMORPHISMS OF GROUPS OF TYPE F
DOI: <https://doi.org/10.5169/seals-61816>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 19.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

$$\chi_1(G)(ht^{vq}) = \left(\sum_{n \geq 0} \sum_{i=0}^{vq-1} (-1)^n A(\text{trace}([\tilde{f}_n][f_n^i])), - (q/r)v \sum_{i=0}^{r-1} L(f^i) \right)$$

and

$$\chi_1(G; \mathbf{Q})(ht^{vq}) = \left(0, - (q/r)v \sum_{i=0}^{r-1} L(f^i) \right) = (q/r)v \sum_{i=0}^{r-1} L(f^i) \{t\}$$

where $h \in \text{Fix}(\theta) \cap h_0^{-vq/r} Z(H)$. \square

Similarly, one can read off formulae for $\tilde{X}_1(G)$ from Theorem 6.14 and the rational version from Theorem 6.16.

8. OUTER AUTOMORPHISMS OF GROUPS OF TYPE \mathcal{F}

In this section we apply the preceding theory to prove the following theorem which relates the algebraic topology of an automorphism $\theta: H \rightarrow H$ of a group H of type \mathcal{F} such that θ has finite order in $\text{Out}(H)$ to the fixed group of θ .

THEOREM 8.1. *Let H be a group of type \mathcal{F} which has the Weak Bass Property over \mathbf{Q} . Suppose that $\theta: H \rightarrow H$ is an automorphism whose order in $\text{Out}(H)$ is $r \geq 1$. If the sum of the Lefschetz numbers $\sum_{i=0}^{r-1} L(\theta^i)$ is non-zero then $Z(H) \cap \text{Fix}(\theta) = (1)$.*

Before proving this we note that the quantity $\sum_{i=0}^{r-1} L(\theta^i)$ appearing above has the following interpretation:

PROPOSITION 8.2. *$\sum_{i=0}^{r-1} L(\theta^i)$ is r times the Euler characteristic of the θ -invariant part of the homology of H , i.e.,*

$$\sum_{i=0}^{r-1} L(\theta^i) = r \sum_{j \geq 0} (-1)^j \text{rank ker}(\text{id} - \theta_j: H_j(H) \rightarrow H_j(H)).$$

Proof. By elementary linear algebra, for any square complex matrix A with $A^r = I$ we have $\text{trace}(\sum_{i=0}^{r-1} A^i) = r \dim \ker(I - A)$. The conclusion easily follows. \square

Proof of Theorem 8.1. Let G be the semidirect product $G = H \rtimes_\theta T$ where T is infinite cyclic. By Lemma 8.7, below, G also has the WBP over \mathbf{Q} . Applying Theorem 7.11 to G , we have that $\chi_1(G; \mathbf{Q}) \neq 0$. By

Theorem 5.4, $Z(G)$ is infinite cyclic. By Corollary 7.9 there is an exact sequence $1 \rightarrow Z(H) \cap \text{Fix}(\theta) \rightarrow Z(G) \xrightarrow{P_*} q\mathbb{Z} \rightarrow 1$ where the period of θ , q , is positive. It follows that $Z(H) \cap \text{Fix}(\theta) = (1)$. \square

If $\chi(H) \neq 0$ then $Z(H) = (1)$ by Proposition 2.4 and consequently $Z(H) \cap \text{Fix}(\theta) = (1)$ in this case. If $\chi(H) = L(\theta^0) = 0$ then $\sum_{i=0}^{r-1} L(\theta^i) = \sum_{i=1}^{r-1} L(\theta^i)$. These observations yield the following corollaries of Theorem 8.1:

COROLLARY 8.3. *Let H be a group of type \mathcal{F} which has the WBP over \mathbf{Q} . Suppose that $\theta: H \rightarrow H$ is an automorphism of order 2 in $\text{Out}(H)$. If $L(\theta) \neq 0$ then $Z(H) \cap \text{Fix}(\theta) = (1)$. \square*

COROLLARY 8.4. *Let H be a group of type \mathcal{F} which has the WBP over \mathbf{Q} . Suppose $Z(H) \neq (1)$, the automorphism $\theta: H \rightarrow H$ has finite order r in $\text{Out}(H)$ and the restriction of θ to $Z(H)$ is the identity. Then $\sum_{i=1}^{r-1} L(\theta^i) = 0$.*

Proof. Since the restriction of θ to $Z(H)$ is the identity, $Z(H) \cap \text{Fix}(\theta) = Z(H) \neq (1)$. \square

An automorphism which has finite order in $\text{Out}(H)$ may have infinite order in $\text{Aut}(H)$. If θ has finite order in $\text{Aut}(H)$, the Weak Bass Property hypothesis can be dispensed with in Theorem 8.1 and Corollary 8.3:

PROPOSITION 8.5. *Let H be a group of type \mathcal{F} . Suppose that $\theta: H \rightarrow H$ has finite order in $\text{Aut}(H)$ and $L(\theta) \neq 0$. Then $Z(H) \cap \text{Fix}(\theta) = (1)$.*

Proof. Let $\omega \in Z(H) \cap \text{Fix}(\theta)$. We use the terminology of [Br]. Let Z be a finite $K(H, 1)$. Choose an essential fixed point, v , of $f: Z \rightarrow Z$ (inducing θ) as the basepoint of Z . There is a homotopy $K: f \simeq f$ such that $K(v, \cdot)$ represents ω . The fixed point v is K -related to some fixed point u of f [Br, p. 92]. Hence, for some $s > 0$, v is J -related to v , where J is the s -fold concatenation $K \star \cdots \star K$. Then there exists $\sigma \in H$ such that $\omega^s = \sigma\theta(\sigma^{-1})$; compare [G]. As in the proof of Proposition 7.7, we get $\omega^{rs} = \prod_{i=0}^{r-1} \theta^i(\sigma\theta(\sigma^{-1})) = 1$, so $\omega = 1$. \square

Note that $\sum_{i=1}^{r-1} L(\theta^i) \neq 0$ implies one of the $L(\theta^i)$'s is non-zero. Since $\text{Fix}(\theta) \subset \text{Fix}(\theta^i)$ for $i \geq 0$, we recover Theorem 8.1 (but without the Bass Conjecture hypothesis) in the special case where θ has finite order in $\text{Aut}(H)$.

The remainder of this section is devoted to the proof of Lemma 8.7 used above.

LEMMA 8.6. *Suppose that the group H has the WBP over \mathbf{Q} . Let T be an infinite cyclic group. Then the product group $H \times T$ also has the WBP over \mathbf{Q} .*

Proof. Let $G = H \times T$. Identify H with $H \times \{1\} \subset G$. We use the notation of §5. By Schafer's theorem [Sch, p. 224] applied to the normal subgroup $H \subset G$, the image of $T_0: K_0(\mathbf{Q}G) \rightarrow HH_0(\mathbf{Q}G)$ lies in $HH_0(\mathbf{Q}G)_H$. Let $p: G \rightarrow H$ be the projection homomorphism. There is a commutative diagram:

$$\begin{array}{ccccc} K_0(\mathbf{Q}G) & \xrightarrow{T_0} & HH_0(\mathbf{Q}G)_H & \xrightarrow{\varepsilon_*} & \mathbf{Q} \\ p_* \downarrow & & p_* \downarrow & & \parallel \\ K_0(\mathbf{Q}H) & \xrightarrow{T_0} & HH_0(\mathbf{Q}H) & \xrightarrow{\varepsilon_*} & \mathbf{Q} \end{array}$$

Write $HH_0(\mathbf{Q}G)_H = HH_0(\mathbf{Q}G)_{C(1)} \oplus HH_0(\mathbf{Q}G)''_H$ where $HH_0(\mathbf{Q}G)''_H$ is the direct sum of the $HH_0(\mathbf{Q}G)_{C(g)}$'s over $C(g) \in c(H) - \{C(1)\}$; also, $HH_0(\mathbf{Q}H) = HH_0(\mathbf{Q}H)_{C(1)} \oplus HH_0(\mathbf{Q}H)'$. By hypothesis, H has the WBP over \mathbf{Q} , i.e. the composite

$$K_0(\mathbf{Q}H) \xrightarrow{T_0} HH_0(\mathbf{Q}H) \rightarrow HH_0(\mathbf{Q}H)' \xrightarrow{\varepsilon_*} \mathbf{Q}$$

is zero. Since $p_*(HH_0(\mathbf{Q}G)_{C(1)}) \subset HH_0(\mathbf{Q}H)_{C(1)}$ and $p_*(HH_0(\mathbf{Q}G)''_H) \subset HH_0(\mathbf{Q}H)'$, the conclusion follows. \square

LEMMA 8.7. *Suppose that the group H has the WBP over \mathbf{Q} and that $\theta: H \rightarrow H$ is an automorphism whose image in the group of outer automorphisms of H has finite order. Then the semidirect product $H \rtimes_\theta T$ also has the WBP over \mathbf{Q} .*

Proof. Let $G = H \rtimes_\theta T \equiv \langle H, t \mid tht^{-1} = \theta(h) \text{ for } h \in H \rangle$. Let n be the order of θ in the group outer automorphisms of H . Then the subgroup G' of G generated by H and t^n is isomorphic to $H \times T$; furthermore, G' is normal and of finite index, n , in G . There is a "transfer" homomorphism $\text{trans}: HH_0(\mathbf{Q}G) \rightarrow HH_0(\mathbf{Q}G')$ defined as follows. Given $g \in G$, we can write $gt^i = t^{\sigma(i)}g_i$ for $i = 0, \dots, n-1$ where $g_i \in G'$ and σ is a permutation of $\{0, \dots, n-1\}$. Let $\text{Fix}(\sigma) = \{i \mid \sigma(i) = i\}$. Then $\text{trans}(C(g)) = \sum_{i \in \text{Fix}(\sigma)} C(g_i)$. Observe that if $g \in G'$ then $\text{Fix}(\sigma) = \{0, \dots, n-1\}$

because G' is normal in G . In particular, $\varepsilon_*(\text{trans}(C(g))) = n$ if $g \in G'$. There is a commutative diagram:

$$\begin{array}{ccc} K_0(\mathbf{Q}G) & \xrightarrow{T_0} & HH_0(\mathbf{Q}G) \\ \text{res} \downarrow & & \text{trans} \downarrow \\ K_0(\mathbf{Q}G') & \xrightarrow{T_0} & HH_0(\mathbf{Q}G') \end{array}$$

where $\text{res}: K_0(\mathbf{Q}G) \rightarrow K_0(\mathbf{Q}G')$ is obtained by regarding a projective $\mathbf{Q}G$ module as a projective $\mathbf{Q}G'$ module; see [Bass] for details concerning the finite index transfer.

Recall that $HH_0(\mathbf{Q}G) = HH_0(\mathbf{Q}G)_H \oplus HH_0(\mathbf{Q}G)'_H$ where $HH_0(\mathbf{Q}G)'_H$ is the direct sum of the summands $HH_0(\mathbf{Q}G)_{C(g)}$ corresponding to the conjugacy classes not represented by elements of H . By Schafer's theorem [Sch, p. 224] applied to the normal subgroup $H \subset G$, the image of $T_0: K_0(\mathbf{Q}G) \rightarrow HH_0(\mathbf{Q}G)$ lies in $HH_0(\mathbf{Q}G)_H$. Thus we can replace $HH_0(\mathbf{Q}G)$ with $HH_0(\mathbf{Q}G)_H$ in the above diagram and obtain the commutative diagram:

$$\begin{array}{ccccc} K_0(\mathbf{Q}G) & \xrightarrow{T_0} & HH_0(\mathbf{Q}G)_H & \xrightarrow{\varepsilon_*} & \mathbf{Q} \\ \text{res} \downarrow & & \text{trans} \downarrow & & \times n \downarrow \\ K_0(\mathbf{Q}G') & \xrightarrow{T_0} & HH_0(\mathbf{Q}G') & \xrightarrow{\varepsilon_*} & \mathbf{Q} \end{array}$$

(the right square commutes because $H \subset G'$ and because of the observation made above). Write $HH_0(\mathbf{Q}G)_H = HH_0(\mathbf{Q}G)_{C(1)} \oplus HH_0(\mathbf{Q}G)''_H$ where $HH_0(\mathbf{Q}G)''_H$ is the direct sum of the $HH_0(\mathbf{Q}G)_{C(g)}$'s over $C(g) \in c(H) - \{C(1)\}$; also, $HH_0(\mathbf{Q}G') = HH_0(\mathbf{Q}G')_{C(1)} \oplus HH_0(\mathbf{Q}G')'$. Then $\text{trans}(HH_0(\mathbf{Q}G)_{C(1)}) \subset HH_0(\mathbf{Q}G')_{C(1)}$ and $\text{trans}(HH_0(\mathbf{Q}G)''_H) \subset HH_0(\mathbf{Q}G')'$. By Lemma 8.6, G' has the WBP over \mathbf{Q} , i.e. the composite $K_0(\mathbf{Q}G') \xrightarrow{T_0} HH_0(\mathbf{Q}G') \rightarrow HH_0(\mathbf{Q}G')' \xrightarrow{\varepsilon_*} \mathbf{Q}$ is zero. The conclusion follows from the above diagram. \square

9. TRACE FORMULAE FOR HOMOLOGICAL INTERSECTIONS

The goal of this section is to prove a "trace formula" (Theorem 9.13) for the homological intersection of the graph of a map $F: M \times Y \rightarrow M$ with the graph of the projection map $p: M \times Y \rightarrow M$ where Y is a closed oriented manifold and M is a compact oriented manifold. This result will be applied in §10 to complete the proof of Theorem 1.1.