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**Autor:** Okonek, Ch. / Van de Ven, A.  
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There will certainly be some gaps for algebraic 3-folds. In order to show this, we prove the following finiteness theorem for families of Kähler structures:

**THEOREM 7.** *Fix a positive constant  $c$ . There exist only finitely many families of 1-connected, smooth projective 3-folds  $X$  with  $H_2(X, \mathbf{Z}) \cong \mathbf{Z}$ ,  $w_2(X) \neq 0$ , and with  $b_3(X) \leq c$ .*

*Proof.* Let  $X$  be a smooth projective 3-fold with  $H_1(X, \mathbf{Z}) = \{0\}$ ,  $H_2(X, \mathbf{Z}) \cong \mathbf{Z}$ , and with  $w_2(X) \neq 0$ . Clearly  $\text{Pic}(X) \cong H^2(X, \mathbf{Z})$ , and we can choose a basis  $e \in H^2(X, \mathbf{Z})$  corresponding to the ample generator of  $\text{Pic}(X)$ .

Let  $c_1(X) = c_1 e$ ,  $c_2(X) = c_2 \varepsilon$  where  $e^2 = d\varepsilon$ ,  $\varepsilon(e) = 1$ . If  $c_1$  is positive, then  $X$  is Fano, and there are only finitely many possibilities [Mu]. The case  $c_1 = 0$  is excluded, so that we are left with  $c_1 < 0$ , i.e. the canonical bundle of  $X$  is ample.

The Riemann-Roch formula  $\chi(X, \mathcal{O}_X) = 1 - h^3(X, \mathcal{O}_X) = \frac{1}{24} c_1 c_2$  shows that the set of possible Chern numbers  $c_1 c_2$  is bounded from below:  $24(1 - c) \leq c_1 c_2$ . Using Yau's inequality  $8c_1(X)c_2(X) \leq 3c_1(X)^3$  we find that  $d |c_1|^3 \leq 64(c - 1)$ , i.e. the degree  $d$  and the order of divisibility  $|c_1|$  of  $c_1(X)$  is bounded. Now Kollar's finiteness theorem [Ko2] yields the assertion.

**EXAMPLE 15.** Let  $X$  be a 1-connected, smooth projective 3-fold with  $H_2(X, \mathbf{Z}) \cong \mathbf{Z}$  and  $w_2(X) \neq 0$ . If  $b_3(X) \leq 2$ , then  $h^3(X, \mathcal{O}_X) \leq 1$  and  $X$  must be Fano of index 1 or 3. For  $b_3(X) = 4$  we have that  $X$  is either Fano, or  $h^3(X, \mathcal{O}_X) = 2$  and  $X$  is of general type with  $d |c_1|^3 \leq 64$ .

Note that the assumption  $w_2 \neq 0$  was only used to exclude Calabi-Yau 3-folds.

## 5.2 3-FOLDS WITH $b_2 = 2$

Let  $X$  be a 1-connected, closed, oriented, 6-dimensional differentiable manifold with  $H_2(X, \mathbf{Z}) \cong \mathbf{Z}^2$ .

We choose a basis  $e_1, e_2$  for  $H^2(X, \mathbf{Z})$  and set  $a_0 = e_1^3$ ,  $a_1 = e_1^2 e_2$ ,  $a_2 = e_1 e_2^2$ ,  $a_3 = e_2^3$ ; the cubic polynomial  $f$  associated to the cup-form of  $X$  is then given by  $f = a_0 X^3 + 3a_1 X^2 Y + 3a_2 X Y^2 + a_3 Y^3$ . The discriminant of  $f$  is by definition  $\Delta(f) = a_0^2 a_3^2 - 3a_1^2 a_2^2 - 6a_0 a_1 a_2 a_3 + 4a_0 a_2^3 + 4a_1^3 a_3$ ; up to a factor it is simply the discriminant of the Hessian  $H_f = 6^2[(a_0 a_2 - a_1^2)X^2 + (a_0 a_3 - a_1 a_2)XY + (a_1 a_3 - a_2^2)Y^2]$  of  $f$ :  $\Delta(f) = (a_0 a_3 - a_1 a_2)^2 - 4(a_0 a_2 - a_1^2)(a_1 a_3 - a_2^2)$ .

The last identity shows that  $\Delta(f)$  is always a square modulo 4, i.e.  $\Delta(f) \equiv 0, 1 \pmod{4}$ .

PROPOSITION 17. *Every integer  $\Delta \equiv 0, 1 \pmod{4}$  is realizable as discriminant of a compact complex 3-fold.*

*Proof.* Consider the projectivization  $X = \mathbf{P}_{\mathbf{P}^2}(E)$  of a holomorphic rank-2 vector bundle  $E$  over the plane. In terms of the standard basis of  $H^2(X, \mathbf{Z})$  ( $e_1 = \pi^*h$ ,  $e_2 = c_1(\mathcal{O}_{\mathbf{P}(E)}(1))$ ) the cubic polynomial associated to  $X$  is given by  $f = (c_1^2 - c_2)X^3 + 3(-c_1)X^2Y + 3XY^2$ , where  $c_i = c_i(E)$  are the Chern classes of  $E$  considered as integers. Inserting this into the discriminant formula yields  $\Delta(f) = c_1^2 - 4c_2$ . Since every pair  $c_1, c_2$  occurs as pair of Chern classes of a holomorphic rank-2 bundle on  $\mathbf{P}^2$ , every integer  $\Delta \equiv 0, 1 \pmod{4}$  can be realized as discriminant of a holomorphic projective bundle  $\mathbf{P}_{\mathbf{P}^2}(E)$ .

Recall from section 3.2 that there are 4 different types of  $SL(2)$ -orbits of complex binary cubics: non-singular forms  $f$  (with  $\Delta(f) \neq 0$ ), and three orbits of singular cubics, represented by the normal forms  $X^2Y$ ,  $X^3$ , and 0.

PROPOSITION 18. *All four types of complex binary cubics are realizable by complex 3-folds.*

*Proof.* We have seen this already for non-singular cubics. Clearly the product  $\mathbf{P}^1 \times \mathbf{P}^2$  realizes the normal form  $X^2Y$ . The cubics of normal forms  $X^3$  or 0 are degenerate, i.e. their Hessians vanish identically. Therefore they can only be realized by non-Kählerian 3-folds. To realize  $X^3$  one can blow up a point in an elliptic fiber bundle over a surface  $Y$  with  $b_2(Y) = 3$ ; the trivial form occurs for elliptic fiber bundles over a surface with  $b_2 = 4$ .

More detailed investigations of the possible homotopy types of real or complex manifolds with  $b_2 = 2$  will appear elsewhere [Sch].

Here we only want to illustrate an interesting phenomenon which relates the ample cone of a projective 3-fold with  $b_2 = 2$  to the Hessian of its cup-form.

PROPOSITION 19. *Let  $X$  be a smooth projective 3-fold with  $b_2(X) = 2$ . The ample cone  $\mathcal{C}_X$  is contained in the Hesse cone  $\mathcal{H}_F := \{h \in H^2(X, \mathbf{R}) \mid \det(F'(h)) < 0\}$ .*

*Proof.* This is only a special case of our general result in section 4.3.

REMARK 14. The Hessian of a binary form  $F \in S^3 H^\vee$  is identically zero iff  $F$  is degenerate; it is negative semi-definite if  $F$  is non-degenerate and  $\Delta(F) \leq 0$ ; it is indefinite iff  $\Delta(F) > 0$  [Ca]. Only in the indefinite case  $\Delta(F) > 0$  can the closure  $\overline{\mathcal{H}}_F := \{h \in H_{\mathbf{R}} \mid \det F'(h) \leq 0\}$  of the Hesse cone be a proper subset of  $H_{\mathbf{R}}$ .

EXAMPLE 16. Let  $P = \mathbf{P}_{\mathbf{P}^2}(E)$  be the projectivization of a rank-2 vector bundle  $E$  with Chern classes  $c_i = c_i(E)$ . The cup-form of  $P$  yields the cubic polynomial  $f = (c_1^2 - c_2)X^2 + 3(-c_1)X^2Y + 3XY^2$  whose Hessian is  $H_f = (-c_2)X^2 + c_1XY - Y^2$ . Rewriting  $H_f$  as  $H_f = -\frac{1}{4}[(2Y - c_1X)^2 + X^2(4c_2 - c_1^2)] = \frac{-1}{4}[(2Y - c_1X)^2 - \Delta(f)X^2]$  we find 3 possibilities for the Hesse cone:

- i)  $\Delta(f) < 0$ :  $\mathcal{H}_f = H^2(P, \mathbf{R}) \setminus \{0\}$
- ii)  $\Delta(f) = 0$ :  $\mathcal{H}_f = H^2(P, \mathbf{R}) \setminus L_{c_1}$  for a real line  $L_{c_1}$  depending on  $c_1$  ( $L_{c_1} = \mathbf{R}(2, c_1)$  in the coordinates  $X, Y$ )
- iii)  $\Delta(f) > 0$ :  $\mathcal{H}_f$  is an open cone whose angle is determined by  $\Delta(f) \left( (Z + \sqrt{\Delta(f)}X)(Z - \sqrt{\Delta(f)}X) > 0 \right)$  in coordinates  $X, Z := 2Y - c_1X$ .

### 5.3 3-FOLDS WITH $b_2 \geq 3$

Let  $X$  be a 1-connected, compact complex 3-fold with  $H_2(X, \mathbf{Z}) \cong \mathbf{Z}^{\oplus 3}$ . The cup-form of  $X$  gives rise to a curve  $C_X$  of degree 3 in the projective plane  $\mathbf{P}(H^2(X, \mathbf{C}))$ :

$$C_X := \{ \langle h \rangle \in \mathbf{P}(H^2(X, \mathbf{C})) \mid h^3 = 0 \}.$$

A first natural question is which types of plane cubic curves occur in this way?

Recall that there are 10 types of plane cubics, namely: 1) non-singular cubics, 2) irreducible cubics with a node, 3) irreducible cubics with a cusp, 4) reducible cubics consisting of a smooth conic and a transversal line, 5) smooth conics with a tangent line, 6) three lines forming a triangle, 7) three distinct lines through a common point, 8) a double line with a third skew line, 9) a triple line, 10) the trivial 'cubic' with equation 0.

LEMMA 4. *If the 3-fold  $X$  has a non-trivial Hodge number  $h^{2,0}(X) \neq 0$ , then  $C_X$  is of type 4), 6) 9) or 10).*