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theorem for 3-folds with $b_2 = 1$, $w_2 \neq 0$, and we give examples which show that the condition $I_{F_X} \neq \emptyset$ for the index cone of a projective 3-fold with $h^{0,2} = 0$ is non-trivial in general.

5.1 3-FOLDS WITH $b_2 = 1$

Recall from section 1.1 that every closed, oriented, 1-connected differentiable 6-manifold X with torsion-free homology has a connected sum decomposition $X \cong X_\circ \#_r S^3 \times S^3$ where $r = \binom{b_3(X)}{2}$, which is unique up to orientation preserving diffeomorphisms; the manifold $X_\circ$ with $b_3(X_\circ) = 0$ is the core of X.

THEOREM 6. *Let $X_\circ$ be a 1-connected, closed, oriented differentiable 6-manifold with $H_2(X_\circ, \mathbf{Z}) \cong \mathbf{Z}$ and $b_3(X_\circ) = 0$. There exists a compact complex 3-fold X whose core is orientation preservingly homotopy equivalent to $X_\circ$.*

Proof. The oriented homotopy type of $X_\circ$ is determined by the invariants d, w_2, and $p_1 \pmod{48}$; more precisely: for $d \equiv 1 \pmod 2$ there is a single homotopy type whereas for $d \equiv 0 \pmod 2$ there are three; one of these 3 types has $w_2 \neq 0$, the other two are spin, they are distinguished by $p_1 \equiv 4d \pmod{48}$, $p_1 \equiv 4d + 24 \pmod{48}$ respectively. In order to realize these homotopy types as cores of complex 3-folds we first look at simple cyclic coverings of $\mathbf{P}^3$. Given a positive integer d, let $\pi : X \to \mathbf{P}^3$ be a simple cyclic covering of $\mathbf{P}^3$ branched along a smooth surface B of degree dl. Then X has the correct 'degree' d and the characteristic classes $w_2 \equiv (d-1)l \pmod 2$, and $p_1 = 4d + (1-d)(1+d)dl^2$, see 4.2. For odd d there is nothing to prove. For even d we can realize $w_2 = 0$ or $w_2 \neq 0$ by choosing $l \equiv 0 \pmod 2$ or $l \equiv 1 \pmod 2$. Taking $l \equiv 0 \pmod 4$ gives $w_2 = 0$, $p_1 \equiv 4d \pmod{48}$, taking $l \equiv 2 \pmod 4$ yields $w_2 = 0$, and $p_1 \equiv 4d + 24 \pmod{48}$. It remains to treat the special case $d = 0$, where the 3 homotopy types are given by $w_2 \neq 0$, by $w_2 = 0$, $p_1 \equiv 0 \pmod{16}$, and by $w_2 = 0$, $p_1 \equiv 8 \pmod{16}$. The first two homotopy types are realizable as cores of elliptic fiber bundles over the projective plane blown up in two points.

The third homotopy type is realized by the core of Oguiso's Calabi-Yau 3-fold X_2 with vanishing cup-form and $p_1(X_2) = 120\varepsilon_2$.

The result just proven suggests a natural question: given a manifold $X_\circ$ as above, which (even) integers $b_3 \geqslant 0$ occur as the third Betti numbers of complex 3-folds X whose core is homotopy equivalent to $X_\circ$?

There will certainly be some gaps for algebraic 3-folds. In order to show this, we prove the following finiteness theorem for families of Kähler structures:

THEOREM 7. *Fix a positive constant* c. *There exist only finitely many families of 1-connected, smooth projective 3-folds* X *with* $H_2(X, \mathbf{Z}) \cong \mathbf{Z}$, $w_2(X) \neq 0$, *and with* $b_3(X) \leqslant c$.

Proof. Let X be a smooth projective 3-fold with $H_1(X, \mathbf{Z}) = \{0\}$, $H_2(X, \mathbf{Z}) \cong \mathbf{Z}$, and with $w_2(X) \neq 0$. Clearly $\mathrm{Pic}(X) \cong H^2(X, \mathbf{Z})$, and we can choose a basis $e \in H^2(X, \mathbf{Z})$ corresponding to the ample generator of $\mathrm{Pic}(X)$.

Let $c_1(X) = c_1 e$, $c_2(X) = c_2 \varepsilon$ where $e^2 = d\varepsilon$, $\varepsilon(e) = 1$. If c_1 is positive, then X is Fano, and there are only finitely many possibilities [Mu]. The case $c_1 = 0$ is excluded, so that we are left with $c_1 < 0$, i.e. the canonical bundle of X is ample.

The Riemann-Roch formula $\chi(X, \mathcal{O}_X) = 1 - h^3(X, \mathcal{O}_X) = \frac{1}{24} c_1 c_2$ shows that the set of possible Chern numbers $c_1 c_2$ is bounded from below: $24(1 - c) \leqslant c_1 c_2$. Using Yau's inequality $8c_1(X)c_2(X) \leqslant 3c_1(X)^3$ we find that $d\,|\,c_1\,|^3 \leqslant 64(c - 1)$, i.e. the degree d and the order of divisibility $|\,c_1\,|$ of $c_1(X)$ is bounded. Now Kollar's finiteness theorem [Ko2] yields the assertion.

EXAMPLE 15. Let X be a 1-connected, smooth projective 3-fold with $H_2(X, \mathbf{Z}) \cong \mathbf{Z}$ and $w_2(X) \neq 0$. If $b_3(X) \leqslant 2$, then $h^3(X, \mathcal{O}_X) \leqslant 1$ and X must be Fano of index 1 or 3. For $b_3(X) = 4$ we have that X is either Fano, or $h^3(X, \mathcal{O}_X) = 2$ and X is of general type with $d\,|\,c_1\,|^3 \leqslant 64$.

Note that the assumption $w_2 \neq 0$ was only used to exclude Calabi-Yau 3-folds.

5.2 3-FOLDS WITH $b_2 = 2$

Let X be a 1-connected, closed, oriented, 6-dimensional differentiable manifold with $H_2(X, \mathbf{Z}) \cong \mathbf{Z}^2$.

We choose a basis e_1, e_2 for $H^2(X, \mathbf{Z})$ and set $a_0 = e_1^3$, $a_1 = e_1^2 e_2$, $a_2 = e_1 e_2^2$, $a_3 = e_2^3$; the cubic polynomial f associated to the cup-form of X is then given by $f = a_0 X^3 + 3a_1 X^2 Y + 3a_2 XY^2 + a_3 Y^3$. The discriminant of f is by definition $\Delta(f) = a_0^2 a_3^2 - 3a_1^2 a_2^2 - 6a_0 a_1 a_2 a_3 + 4a_0 a_2^3 + 4a_1^3 a_3$; up to a factor it is simply the discriminant of the Hessian $H_f = 6^2[(a_0 a_2 - a_1^2)X^2 + (a_0 a_3 - a_1 a_2)XY + (a_1 a_3 - a_2^2)Y^2]$ of f: $\Delta(f) = (a_0 a_3 - a_1 a_2)^2 - 4(a_0 a_2 - a_1^2)(a_1 a_3 - a_2^2)$.