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REMARK 12. It is very likely that there exist non-integrable almost complex structures on manifolds X as above, but probably this is hard to prove. It is also not unlikely that the Chern numbers of complex 3-folds are not topological invariants. A possible way to check this would be, to run a computer search for 3-folds given by certain standard constructions.

4.2 STANDARD CONSTRUCTIONS

For later use we investigate the topological invariants of complex 3-folds which can be obtained by certain simple standard constructions like complete intersections, simple cyclic coverings, blow-ups of points and curves, and projective bundles.

PROPOSITION 11 (Libgober/Wood). *Let* $X \subset \mathbf{P}^{3+r}$ *be a smooth complete intersection of multidegree* $\underline{d} = (d_1, \ldots, d_r)$. *Choose a normalized basis* $e \in H^2(X, \mathbf{Z})$, *and let* $\varepsilon \in H^4(X, \mathbf{Z})$ *be defined by* $\varepsilon(e) = 1$. *Then the invariants of* X *are:*

$F_X(xe) = dx^3$ *where* $d = \prod_{i=1}^r d_i$, $w_2(X) \equiv (4 + r - \sum_{i=1}^r d_i)e$,

$p_1(X) = d(4 + r - \sum_{i=1}^r d_i^2)\varepsilon$, *and*

$$b_3(X) = 4 - \tfrac{d}{6}[(4 + r - \textstyle\sum_{i=1}^r d_i)^3 - 3(4 + r - \sum_{i=1}^r d_i)(4 + r - \sum_{i=1}^r d_i^2) + 2(4 + r - \sum_{i=1}^r d_i^3)] .$$

Proof. [L/W].

PROPOSITION 12. *Let* X *be a smooth, 1-connected, complex projective 3-fold, and let* $\pi : X' \to X$ *be a simple cyclic covering of degree* d *branched along a non-singular ample divisor* $B \in |L^{\otimes d}|$. X' *is smooth, projective, 1-connected, and* $\pi^* : H^2(X, \mathbf{Z}) \to H^2(X', \mathbf{Z})$ *is an isomorphism. The invariants of* X *and* X' *are related by the formulae:*

$$(\pi^*)^* F_{X'} = dF_X, \quad w_2(X') - \pi^* w_2(X) \equiv (d-1)\pi^* c_1(L),$$

$$p_1(X') - \pi^* p_1(X) = (1-d)(1+d)\pi^* c_1(L)^2, \quad \textit{and}$$

$$b_3(X') = db_3(X) + (d-1)(b_2(B) - 2b_2(X)) .$$

Proof. X' is clearly smooth and projective. By a theorem of M. Cornalba $\pi : X' \to X$ is a 3-equivalence, i.e. $\pi_* : \pi_i(X') \to \pi_i(X)$ is bijective for $i \leqslant 2$, and surjective for $i = 3$ [*Co*]. X' is therefore 1-connected, and $\pi^* : H^2(X, \mathbf{Z}) \to H^2(X', \mathbf{Z})$ is an isomorphism. The relation between $F_{X'}$ and F_X is obvious, whereas the formula for $b_3(X')$ follows from $\pi_1(B) = \{1\}$ and standard properties of Euler numbers.

In order to calculate $w_2(X')$ and $p_1(X')$ we compute the Chern classes of $X': c_1(X') - \pi^* c_1(X) = (1-d)\pi^* c_1(L), c_2(X') - \pi^* c_2(X) = (1-d)\pi^*[c_1(X)c_1(L) - dc_1(L)^2]$.

The latter formulae follow from the description of X' as a divisor in the total space of the line bundle L.

EXAMPLE 9. Let X be a d-fold, simple cyclic covering of $\mathbf{P}^3$ branched along a smooth surface $B \subset \mathbf{P}^3$ of degree $dl, l \geqslant 1$. Let $e \in H^2(X, \mathbf{Z})$ correspond to the preimage of a plane in $\mathbf{P}^3$. The invariants of X are then given by:

$$F_X(xe) = dx^3,\ w_2(X) \equiv (4 + (1-d)l)e, p_1(X) = d[4 + (1-d)(1+d)l^2]\varepsilon$$
$$(\varepsilon(e) = 1), b_3(X) = (d-1)(d^2l^2 - 4dl + 6)dl\ .$$

PROPOSITION 13. *Let $\sigma: \hat{X} \to X$ be the blow-up of a complex 3-fold X in a point, and let $e \in H^2(\hat{X}, \mathbf{Z})$ be the class of the exceptional divisor. The invariants of $\hat{X}$ and X are related by the following formulae:*

$$F_{\hat{X}}(\sigma^* h + xe) = F_X(h) + x^3\ \forall\ h \in H^2(X, \mathbf{Z}), x \in \mathbf{Z},\ w_2(\hat{X}) = \sigma^* w_2(X),$$
$$p_1(\hat{X}) = \sigma^* p_1(X) + 4(e^2 - \sigma^* c_1(X)\cdot e),\ b_3(\hat{X}) = b_3(X)\ .$$

Proof. Standard arguments, see [G/H]. The Chern classes are related by $c_1(\hat{X}) = \sigma^* c_1(X) - 2e, c_2(\hat{X}) = \sigma^* c_2(X)$.

PROPOSITION 14. *Let $\sigma: \hat{X} \to X$ be the blow-up of a complex 3-fold X along a smooth curve C of genus g, and let $e \in H^2(\hat{X}, \mathbf{Z})$ be the class of the exceptional divisor. The invariants of $\hat{X}$ and X are related by:*

$$F_{\hat{X}}(\sigma^* h + xe) = F_X(h) - 3h\cdot Cx^2 - \deg N_{C/X} x^3\ \forall\ h \in H^2(X, \mathbf{Z}),$$
$$x \in \mathbf{Z}, w_2(\hat{X}) \equiv \sigma^* w_2(X) + e, p_1(\hat{X}) = \sigma^* p_1(X) + (e^2 - 2\sigma^* C),$$
$$b_3(\hat{X}) = b_3(X) + 2g\ .$$

Proof. [G/H]. The Chern classes are given by $c_1(\hat{X}) = \sigma^* c_1(X) - c, c_2(\hat{X}) = \sigma^*(c_2(X) + C) - \sigma^* c_1(X)\cdot e$.

PROPOSITION 15. *Let E be a holomorphic vector bundle of rank 2 with Chern classes $c_i(E), i = 1, 2$ over a 1-connected, compact complex surface Y, and let $\pi: \mathbf{P}(E) \to Y$ be the projective bundle of lines in the fibers of E. The cup-form of $\mathbf{P}(E)$ is given by*

$$F_{\mathbf{P}(E)}(h + x\xi) = x[(3h^2) - (3c_1(E)\cdot h)x + (c_1(E)^2 - c_2(E))x^2]\ ,$$

where $\xi = c_1(\mathcal{O}_{\mathbf{P}(E)}(1))$, $h \in H^2(Y, \mathbf{Z})$, *and* $x \in \mathbf{Z}$. *The other topological invariants of* $\mathbf{P}(E)$ *are:*

$$w_2(\mathbf{P}(E)) \equiv \pi^*(w_2(Y) + c_1(E)), p_1(E))$$
$$= \pi^*[c_1(Y)^2 - 2c_2(Y) + c_1(E)^2 - 4c_2(E)],\ b_3(\mathbf{P}(E)) = 0\ .$$

Proof. The Leray-Hirsch theorem identifies the cohomology ring $H^*(\mathbf{P}(E), \mathbf{Z})$ with the ring $H^*(Y, \mathbf{Z})[\xi]/_{<\xi^2 + c_1(E)\cdot\xi + c_2(E)>}$; this determines the cup-form. In order to calculate the characteristic classes one uses the exact sequence $0 \to \mathcal{O}_{\mathbf{P}(E)} \to \pi^*E \otimes \mathcal{O}_{\mathbf{P}(E)}(1) \to T_{\mathbf{P}(E)} \to \pi^*T_Y \to 0$. $b_3(\mathbf{P}(E)) = 0$ follows from $b_1(Y) = 0$ and the Leray-Hirsch theorem.

4.3 Examples of 1-connected non-Kählerian 3-folds

Recall that the Hessian of a symmetric trilinear form $F \in S^3H^\vee$ on a free $\mathbf{Z}$-module H of finite rank was defined as the composition $H_F: H \overset{F^t}{\to} S^2H^\vee \overset{\text{disc}}{\to} \mathbf{Z}$. In terms of coordinates $\xi_1, \ldots, \xi_b$ on H it is given by the determinant $\det\left(\frac{\partial^2 f}{\partial\xi_i\partial\xi_j}\right)$, where $f \in \mathbf{C}[H_{\mathbf{C}}]_3$ is the homogeneous cubic polynomial associated with F.

PROPOSITION 16. *Let* F *be a symmetric trilinear form whose Hessian vanishes identically. Then* F *is not realizable as cup-form of a Kählerian 3-fold.*

Proof. Let X be a complex 3-fold with a Kähler metric g. The Kähler class $[\omega_g] \in H^2(X, \mathbf{R})$ defines a multiplication map $\cdot[\omega_g]: H^2(X, \mathbf{R}) \to H^4(X, \mathbf{R})$, which is an isomorphism by the Hard Lefschetz Theorem [G/H]. In section 3.1 we have seen that this is not possible if the Hessian of the cup-form vanishes.

COROLLARY 6. *Cubic forms* $f \in \mathbf{C}[H_{\mathbf{C}}]_3$ *which depend on strictly less than* $b = rk_{\mathbf{Z}}H$ *variables are not realizable as cup-forms of Kählerian 3-folds with* $b_2 = b$.

By considering the Hessian of a cup-form over the reals one obtains further conditions.

DEFINITION 4. *Let* $F \in S^3H^\vee$ *be a symmetric trilinear form on a free* $\mathbf{Z}$*-module of rank* b.

The Hesse cone of F *is the subset* $\mathcal{H}_F \subset H_{\mathbf{R}}$ *defined by* $\mathcal{H}_F := \{h \in H_{\mathbf{R}} \mid (-1)^b \det(F^t(h)) < 0\}$.