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REMARK 12. It is very likely that there exist non-integrable almost complex structures on manifolds  $X$  as above, but probably this is hard to prove. It is also not unlikely that the Chern numbers of complex 3-folds are not topological invariants. A possible way to check this would be, to run a computer search for 3-folds given by certain standard constructions.

## 4.2 STANDARD CONSTRUCTIONS

For later use we investigate the topological invariants of complex 3-folds which can be obtained by certain simple standard constructions like complete intersections, simple cyclic coverings, blow-ups of points and curves, and projective bundles.

PROPOSITION 11 (Libgober/Wood). *Let  $X \subset \mathbf{P}^{3+r}$  be a smooth complete intersection of multidegree  $\underline{d} = (d_1, \dots, d_r)$ . Choose a normalized basis  $e \in H^2(X, \mathbf{Z})$ , and let  $\varepsilon \in H^4(X, \mathbf{Z})$  be defined by  $\varepsilon(e) = 1$ . Then the invariants of  $X$  are:*

$$F_X(xe) = dx^3 \quad \text{where } d = \prod_{i=1}^r d_i, \quad w_2(X) \equiv (4 + r - \sum_{i=1}^r d_i)e,$$

$$p_1(X) = d(4 + r - \sum_{i=1}^r d_i^2)\varepsilon, \quad \text{and}$$

$$b_3(X) = 4 - \frac{d}{6} [(4 + r - \sum_{i=1}^r d_i)^3 - 3(4 + r - \sum_{i=1}^r d_i)(4 + r - \sum_{i=1}^r d_i^2) + 2(4 + r - \sum_{i=1}^r d_i^3)].$$

*Proof.* [L/W].

PROPOSITION 12. *Let  $X$  be a smooth, 1-connected, complex projective 3-fold, and let  $\pi: X' \rightarrow X$  be a simple cyclic covering of degree  $d$  branched along a non-singular ample divisor  $B \in |L^{\otimes d}|$ .  $X'$  is smooth, projective, 1-connected, and  $\pi^*: H^2(X, \mathbf{Z}) \rightarrow H^2(X', \mathbf{Z})$  is an isomorphism. The invariants of  $X$  and  $X'$  are related by the formulae:*

$$(\pi^*)^* F_{X'} = dF_X, \quad w_2(X') - \pi^* w_2(X) \equiv (d-1)\pi^* c_1(L),$$

$$p_1(X') - \pi^* p_1(X) = (1-d)(1+d)\pi^* c_1(L)^2, \quad \text{and}$$

$$b_3(X') = db_3(X) + (d-1)(b_2(B) - 2b_2(X)).$$

*Proof.*  $X'$  is clearly smooth and projective. By a theorem of M. Cornalba  $\pi: X' \rightarrow X$  is a 3-equivalence, i.e.  $\pi_*: \pi_i(X') \rightarrow \pi_i(X)$  is bijective for  $i \leq 2$ , and surjective for  $i = 3$  [Co].  $X'$  is therefore 1-connected, and  $\pi^*: H^2(X, \mathbf{Z}) \rightarrow H^2(X', \mathbf{Z})$  is an isomorphism. The relation between  $F_{X'}$  and  $F_X$  is obvious, whereas the formula for  $b_3(X')$  follows from  $\pi_1(B) = \{1\}$  and standard properties of Euler numbers.

In order to calculate  $w_2(X')$  and  $p_1(X')$  we compute the Chern classes of  $X'$ :  $c_1(X') - \pi^*c_1(X) = (1 - d)\pi^*c_1(L)$ ,  $c_2(X') - \pi^*c_2(X) = (1 - d)\pi^*[c_1(X)c_1(L) - dc_1(L)^2]$ .

The latter formulae follow from the description of  $X'$  as a divisor in the total space of the line bundle  $L$ .

EXAMPLE 9. Let  $X$  be a  $d$ -fold, simple cyclic covering of  $\mathbf{P}^3$  branched along a smooth surface  $B \subset \mathbf{P}^3$  of degree  $dl, l \geq 1$ . Let  $e \in H^2(X, \mathbf{Z})$  correspond to the preimage of a plane in  $\mathbf{P}^3$ . The invariants of  $X$  are then given by:

$$F_X(xe) = dx^3, w_2(X) \equiv (4 + (1 - d)l)e, p_1(X) = d[4 + (1 - d)(1 + d)l^2]\varepsilon$$

$$(\varepsilon(e) = 1), b_3(X) = (d - 1)(d^2l^2 - 4dl + 6)dl.$$

PROPOSITION 13. Let  $\sigma: \hat{X} \rightarrow X$  be the blow-up of a complex 3-fold  $X$  in a point, and let  $e \in H^2(\hat{X}, \mathbf{Z})$  be the class of the exceptional divisor. The invariants of  $\hat{X}$  and  $X$  are related by the following formulae:

$$F_{\hat{X}}(\sigma^*h + xe) = F_X(h) + x^3 \quad \forall h \in H^2(X, \mathbf{Z}), x \in \mathbf{Z}, w_2(\hat{X}) = \sigma^*w_2(X),$$

$$p_1(\hat{X}) = \sigma^*p_1(X) + 4(e^2 - \sigma^*c_1(X) \cdot e), b_3(\hat{X}) = b_3(X).$$

*Proof.* Standard arguments, see [G/H]. The Chern classes are related by  $c_1(\hat{X}) = \sigma^*c_1(X) - 2e, c_2(\hat{X}) = \sigma^*c_2(X)$ .

PROPOSITION 14. Let  $\sigma: \hat{X} \rightarrow X$  be the blow-up of a complex 3-fold  $X$  along a smooth curve  $C$  of genus  $g$ , and let  $e \in H^2(\hat{X}, \mathbf{Z})$  be the class of the exceptional divisor. The invariants of  $\hat{X}$  and  $X$  are related by:

$$F_{\hat{X}}(\sigma^*h + xe) = F_X(h) - 3h \cdot Cx^2 - \text{deg}N_{C/X}x^3 \quad \forall h \in H^2(X, \mathbf{Z}),$$

$$x \in \mathbf{Z}, w_2(\hat{X}) \equiv \sigma^*w_2(X) + e, p_1(\hat{X}) = \sigma^*p_1(X) + (e^2 - 2\sigma^*C),$$

$$b_3(\hat{X}) = b_3(X) + 2g.$$

*Proof.* [G/H]. The Chern classes are given by  $c_1(\hat{X}) = \sigma^*c_1(X) - c, c_2(\hat{X}) = \sigma^*(c_2(X) + C) - \sigma^*c_1(X) \cdot e$ .

PROPOSITION 15. Let  $E$  be a holomorphic vector bundle of rank 2 with Chern classes  $c_i(E), i = 1, 2$  over a 1-connected, compact complex surface  $Y$ , and let  $\pi: \mathbf{P}(E) \rightarrow Y$  be the projective bundle of lines in the fibers of  $E$ . The cup-form of  $\mathbf{P}(E)$  is given by

$$F_{\mathbf{P}(E)}(h + x\xi) = x[(3h^2) - (3c_1(E) \cdot h)x + (c_1(E)^2 - c_2(E))x^2],$$

where  $\xi = c_1(\mathcal{O}_{\mathbf{P}(E)}(1))$ ,  $h \in H^2(Y, \mathbf{Z})$ , and  $x \in \mathbf{Z}$ . The other topological invariants of  $\mathbf{P}(E)$  are:

$$\begin{aligned} w_2(\mathbf{P}(E)) &\equiv \pi^*(w_2(Y) + c_1(E)), p_1(E)) \\ &= \pi^*[c_1(Y)^2 - 2c_2(Y) + c_1(E)^2 - 4c_2(E)], b_3(\mathbf{P}(E)) = 0. \end{aligned}$$

*Proof.* The Leray-Hirsch theorem identifies the cohomology ring  $H^*(\mathbf{P}(E), \mathbf{Z})$  with the ring  $H^*(Y, \mathbf{Z})[\xi]/\langle \xi^2 + c_1(E) \cdot \xi + c_2(E) \rangle$ ; this determines the cup-form. In order to calculate the characteristic classes one uses the exact sequence  $0 \rightarrow \mathcal{O}_{\mathbf{P}(E)} \rightarrow \pi^*E \otimes \mathcal{O}_{\mathbf{P}(E)}(1) \rightarrow T_{\mathbf{P}(E)} \rightarrow \pi^*T_Y \rightarrow 0$ .  $b_3(\mathbf{P}(E)) = 0$  follows from  $b_1(Y) = 0$  and the Leray-Hirsch theorem.

### 4.3 EXAMPLES OF 1-CONNECTED NON-KÄHLERIAN 3-FOLDS

Recall that the Hessian of a symmetric trilinear form  $F \in S^3H^\vee$  on a free  $\mathbf{Z}$ -module  $H$  of finite rank was defined as the composition  $H_F: H \xrightarrow{F^t} S^2H^\vee \xrightarrow{\text{disc}} \mathbf{Z}$ . In terms of coordinates  $\xi_1, \dots, \xi_b$  on  $H$  it is given by the determinant  $\det\left(\frac{\partial^2 f}{\partial \xi_i \partial \xi_j}\right)$ , where  $f \in \mathbf{C}[H_{\mathbf{C}}]_3$  is the homogeneous cubic polynomial associated with  $F$ .

**PROPOSITION 16.** *Let  $F$  be a symmetric trilinear form whose Hessian vanishes identically. Then  $F$  is not realizable as cup-form of a Kählerian 3-fold.*

*Proof.* Let  $X$  be a complex 3-fold with a Kähler metric  $g$ . The Kähler class  $[\omega_g] \in H^2(X, \mathbf{R})$  defines a multiplication map  $\cdot [\omega_g]: H^2(X, \mathbf{R}) \rightarrow H^4(X, \mathbf{R})$ , which is an isomorphism by the Hard Lefschetz Theorem [G/H]. In section 3.1 we have seen that this is not possible if the Hessian of the cup-form vanishes.

**COROLLARY 6.** *Cubic forms  $f \in \mathbf{C}[H_{\mathbf{C}}]_3$  which depend on strictly less than  $b = \text{rk}_{\mathbf{Z}}H$  variables are not realizable as cup-forms of Kählerian 3-folds with  $b_2 = b$ .*

By considering the Hessian of a cup-form over the reals one obtains further conditions.

**DEFINITION 4.** *Let  $F \in S^3H^\vee$  be a symmetric trilinear form on a free  $\mathbf{Z}$ -module of rank  $b$ .*

*The Hesse cone of  $F$  is the subset  $\mathcal{H}_F \subset H_{\mathbf{R}}$  defined by  $\mathcal{H}_F := \{h \in H_{\mathbf{R}} \mid (-1)^b \det(F^t(h)) < 0\}$ .*