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**Autor:** Okonek, Ch. / Van de Ven, A.  
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The number of integral classes in these orbits is therefore finite. We have, however, an even stronger finiteness theorem for stable ternary cubics:

**PROPOSITION 7.** *Let  $H$  be a free  $\mathbf{Z}$ -module of rank 3. There exist only finitely many classes of symmetric trilinear forms  $F \in S^3 H^\vee$  with a fixed discriminant  $\Delta \neq 0$ .*

*Proof.* In terms of Arnhold's invariants  $S$  and  $T$ ,  $\Delta$  is given by  $\Delta = S^3 - T^2$ . By a theorem of C. Siegel [Si], the diophantine equation  $S^3 - T^2 = \Delta$  has only finitely many integral solution  $(S, T)$  for any integer  $\Delta \neq 0$ . For each of these solutions the corresponding point in  $S^3 H_C^\vee / SL(H_C)$  lies outside of the discriminant curve, so that the  $\pi$ -fiber over it is a closed  $SL(H_C)$ -orbit. The finiteness of the class number then follows from the Borel/Harish-Chandra theorem.

A famous special case of Siegel's theorem is Bachet's equation  $S^3 - T^2 = 2$ ; it has only the two obvious solutions  $(3, \pm 5)$ .

**REMARK 10.** To get finiteness results for ternary cubic forms it is not sufficient to fix the  $J$ -invariant (instead of the discriminant): The forms  $f_m = X^3 + XZ^2 + Z^3 + mY^2Z$ ,  $m \in \mathbf{Z} \setminus \{0\}$ , all have the same  $J$ -invariant, but they are not equivalent, even over  $\mathbf{Q}$ , since they have bad reduction at different primes  $p \mid m$ .

#### 4. INVARIANTS OF COMPLEX 3-FOLDS

In this section we begin to investigate the topology of 1-connected, compact, complex 3-folds. After a brief discussion of the possible systems of Chern numbers of almost complex 6-manifolds, we study the behaviour of the topological invariants of complex 3-folds under certain standard constructions, like e.g. branched coverings, or blow-ups of points and curves. Then we describe some interesting examples of 1-connected, non-Kählerian 3-folds, including a new construction method which generalizes the Calabi-Eckmann manifolds. These examples will be needed in the next section in order to realize complex types of cubic forms as cup-forms of complex 3-folds.

##### 4.1 CHERN NUMBERS OF ALMOST COMPLEX STRUCTURES

Let  $X$  be a closed, oriented, 6-dimensional differentiable manifold. The tangent bundle of  $X$  is induced by a classifying map  $t_X: X \rightarrow BSO(6)$  which is unique up to homotopy. By an almost complex structure on  $X$  we mean the homotopy class  $[\tilde{t}_X]$  of a lifting  $\tilde{t}_X: X \rightarrow BU(3)$  of  $t_X$  to  $BU(3)$ .

PROPOSITION 8. *Every closed, oriented, 6-dimensional  $C^\infty$ -manifold  $X$  without 2-torsion in  $H^3(X, \mathbf{Z})$  admits an almost complex structure. There is a 1-1 correspondence between almost complex structures on  $X$  and integral lifts  $W \in H^2(X, \mathbf{Z})$  of  $w_2(X)$ . The Chern classes  $c_i$  of the almost complex manifold  $(X, W)$  are given by  $c_1 = W$ ,  $c_2 = \frac{1}{2}(W^2 - p_1(X))$ .*

*Proof* (cf. [W]). The obstructions against lifting  $t_X$  to  $BU(3)$  lie in the cohomology groups  $H^{i+1}(X, \pi_i(SO(6)/_{U(3)}))$ ,  $i = 0, 1, \dots, 5$ . Since  $SO(6)/_{U(3)} = \mathbf{P}^3$  has only one nontrivial homotopy group  $\pi_2(SO(6)/_{U(3)}) \cong \mathbf{Z}$  in dimensions  $i \leq 5$ , there is in fact only one obstruction  $o(t_X) \in H^3(X, \mathbf{Z})$ , and this obstruction can be identified with the image of  $w_2(X)$  under the Bockstein homomorphism  $\beta: H^2(X, \mathbf{Z}/_2) \rightarrow H^3(X, \mathbf{Z})$ . Since  $H^3(X, \mathbf{Z})$  has no 2-torsion by assumption,  $\beta w_2(X)$  must be equal to zero, so that  $X$  has at least one almost complex structure  $[\tilde{t}_X] \in [X, BU(3)]$ . Standard homotopy arguments show now that the map, which assigns to an almost complex structure  $[\tilde{t}_X]$  its first Chern class  $\tilde{t}_X^* c_1$ , induces a 1-1 correspondence between integral lifts  $W \in H^2(X, \mathbf{Z})$  of  $w_2(X)$  and homotopy classes of liftings of  $[t_X]$  to  $BU(3)$ .

The second Chern class  $c_2$  of the almost complex manifold  $(X, W)$  is determined by  $W^2 - 2c_2 = p_1(X)$ .

The Chern numbers  $c_1^3, c_1 c_2, c_3$  of an almost complex manifold  $X$  of real dimension 6 satisfy the following congruences:  $c_1^3 \equiv 0 \pmod{2}$ ,  $c_1 c_2 \equiv 0 \pmod{24}$ ,  $c_3 \equiv 0 \pmod{2}$ . Conversely, given a triple  $(a, b, c)$  of integers  $a \equiv 0 \pmod{2}$ ,  $b \equiv 0 \pmod{24}$ , and  $c \equiv 0 \pmod{2}$ , there always exists an almost complex manifold  $X$  of dimension 6 with Chern numbers  $c_1^3 = a$ ,  $c_1 c_2 = b$ ,  $c_3 = c$ .

It is not totally clear, however, that one can find a *connected* manifold  $X$  with prescribed Chern numbers [H1].

PROPOSITION 9. *Every triple  $(a, b, c) \in \mathbf{Z}^{\oplus 3}$  satisfying  $a \equiv 0 \pmod{2}$ ,  $b \equiv 0 \pmod{24}$ ,  $c \equiv 0 \pmod{2}$  is realizable as the Chern numbers of an almost complex 6-manifold.*

*Proof.* Consider the complete intersection  $V(f, g) \subset \mathbf{P}^5$  defined by the polynomials  $f(z) = z_0^2 + z_1^2 + 2z_2^2 - z_3^2 - z_4^2 - 2z_5^2$ , and  $g(z) = z_0^4 + z_1^4 + 2z_2^4 - z_3^4 - z_4^4 - 2z_5^4$  [We].  $V(f, g)$  is a singular 3-fold with 90 ordinary double points, and every small resolution  $V$  of these nodes is a (not necessarily projective) Calabi-Yau 3-fold with Euler number 4. Suppose now that a prescribed triple  $(a, b, c) \in \mathbf{Z}^{\oplus 3}$  is realized by a possibly disconnected almost complex manifold  $X = \coprod_{i \in I} X_i$ . If we form the connected sum

$X'$  of the  $X_i$ , we obtain a connected almost complex manifold  $X'$  with Chern numbers  $c_1^3 = a$ ,  $c_1 c_2 = b$ , but with  $c_3 = c - 2(|I| - 1)$ .

If  $|I| > 1$  take the connected sum of  $X'$  with  $|I| - 1$  copies of the complex manifold  $V$ . Since  $V$  is Calabi-Yau, the Chern numbers  $c_1^3$  and  $c_1 c_2$  remain unchanged, whereas the Euler number of  $X' \#_{|I|-1} V$  becomes  $c_3 = c$ .

REMARK 11. The above argument has been suggested by F. Hirzebruch after talk at the MPI, in which one of us had sketched a less geometric proof of the proposition.

There is another question which is related to the result above: Fix a closed, oriented, 6-dimensional differentiable manifold  $X$ . Which pairs  $(a, b)$  of integers with  $a \equiv 0 \pmod{2}$  and  $b \equiv 0 \pmod{24}$  occur as Chern numbers  $c_1^3$  and  $c_1 c_2$  of almost complex structures on  $X$ , and in how many ways?

For manifolds with  $b_2(X) = 1$  the Chern numbers determine the almost complex structure. For manifolds with  $b_2 > 1$  this is no longer true. It is possible to construct infinitely many distinct almost complex structures with the same Chern numbers on a hypersurface of bidegree  $(3, 3)$  in  $\mathbf{P}^2 \times \mathbf{P}^2$ .

An almost complex structure  $[\tilde{t}_X]$  on a differentiable 6-manifold  $X$  is said to be integrable if  $\tilde{t}_X$  is homotopic to the classifying map of a complex 3-fold. We are not aware of any example of an almost complex 6-manifold which is known not to be integrable. On the other hand, it is also unknown whether or not the Chern numbers  $c_1^3, c_1 c_2$  of integrable almost complex manifold are topological invariants. The following remark might therefore be of some interest:

PROPOSITION 10. *If the Chern numbers of complex 3-folds are topological invariants, then there exist almost complex structures which are not integrable.*

*Proof.* Consider a closed, oriented differentiable 6-manifold  $X$  without 2-torsion in  $H^3(X, \mathbf{Z})$ . Fix any almost complex structure on  $X$  with first Chern class  $W \in H^2(X, \mathbf{Z})$ .

Every element  $x \in H^2(X, \mathbf{Z})$  defines a new almost complex structure on  $X$  with first Chern class  $W + 2x$ , and it is easy to see that these two almost complex structures have the same Chern numbers if and only if  $x$  satisfies the equations  $p_1(X) \cdot x = 0$ , and  $3W^2 \cdot x + 6W \cdot x^2 + 4x^3 = 0$ .

Suppose now  $(X, W)$  is integrable,  $p_1(X) \neq 0$ , and choose  $x \in H^2(X, \mathbf{Z})$  such that  $p_1(X) \cdot x \neq 0$ . Then clearly, either none of the almost complex manifolds  $(X, W + 2x)$  is integrable, or the Chern numbers of complex 3-folds are not topologically invariant.

REMARK 12. It is very likely that there exist non-integrable almost complex structures on manifolds  $X$  as above, but probably this is hard to prove. It is also not unlikely that the Chern numbers of complex 3-folds are not topological invariants. A possible way to check this would be, to run a computer search for 3-folds given by certain standard constructions.

## 4.2 STANDARD CONSTRUCTIONS

For later use we investigate the topological invariants of complex 3-folds which can be obtained by certain simple standard constructions like complete intersections, simple cyclic coverings, blow-ups of points and curves, and projective bundles.

PROPOSITION 11 (Libgober/Wood). *Let  $X \subset \mathbf{P}^{3+r}$  be a smooth complete intersection of multidegree  $\underline{d} = (d_1, \dots, d_r)$ . Choose a normalized basis  $e \in H^2(X, \mathbf{Z})$ , and let  $\varepsilon \in H^4(X, \mathbf{Z})$  be defined by  $\varepsilon(e) = 1$ . Then the invariants of  $X$  are:*

$$\begin{aligned} F_X(xe) &= dx^3 \quad \text{where } d = \prod_{i=1}^r d_i, \quad w_2(X) \equiv (4 + r - \sum_{i=1}^r d_i)e, \\ p_1(X) &= d(4 + r - \sum_{i=1}^r d_i^2)\varepsilon, \quad \text{and} \\ b_3(X) &= 4 - \frac{d}{6} \left[ (4 + r - \sum_{i=1}^r d_i)^3 - 3(4 + r - \sum_{i=1}^r d_i)(4 + r - \sum_{i=1}^r d_i^2) \right. \\ &\quad \left. + 2(4 + r - \sum_{i=1}^r d_i^3) \right]. \end{aligned}$$

*Proof.* [L/W].

PROPOSITION 12. *Let  $X$  be a smooth, 1-connected, complex projective 3-fold, and let  $\pi: X' \rightarrow X$  be a simple cyclic covering of degree  $d$  branched along a non-singular ample divisor  $B \in |L^{\otimes d}|$ .  $X'$  is smooth, projective, 1-connected, and  $\pi^*: H^2(X, \mathbf{Z}) \rightarrow H^2(X', \mathbf{Z})$  is an isomorphism. The invariants of  $X$  and  $X'$  are related by the formulae:*

$$\begin{aligned} (\pi^*)^* F_{X'} &= dF_X, \quad w_2(X') - \pi^* w_2(X) \equiv (d-1)\pi^* c_1(L), \\ p_1(X') - \pi^* p_1(X) &= (1-d)(1+d)\pi^* c_1(L)^2, \quad \text{and} \\ b_3(X') &= db_3(X) + (d-1)(b_2(B) - 2b_2(X)). \end{aligned}$$

*Proof.*  $X'$  is clearly smooth and projective. By a theorem of M. Cornalba  $\pi: X' \rightarrow X$  is a 3-equivalence, i.e.  $\pi_*: \pi_i(X') \rightarrow \pi_i(X)$  is bijective for  $i \leq 2$ , and surjective for  $i = 3$  [Co].  $X'$  is therefore 1-connected, and  $\pi^*: H^2(X, \mathbf{Z}) \rightarrow H^2(X', \mathbf{Z})$  is an isomorphism. The relation between  $F_{X'}$  and  $F_X$  is obvious, whereas the formula for  $b_3(X')$  follows from  $\pi_1(B) = \{1\}$  and standard properties of Euler numbers.