

## 4. Invariants of complex 3-folds

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The number of integral classes in these orbits is therefore finite. We have, however, an even stronger finiteness theorem for stable ternary cubics:

**PROPOSITION 7.** *Let  $H$  be a free  $\mathbf{Z}$ -module of rank 3. There exist only finitely many classes of symmetric trilinear forms  $F \in S^3 H^\vee$  with a fixed discriminant  $\Delta \neq 0$ .*

*Proof.* In terms of Arnhold's invariants  $S$  and  $T$ ,  $\Delta$  is given by  $\Delta = S^3 - T^2$ . By a theorem of C. Siegel [Si], the diophantine equation  $S^3 - T^2 = \Delta$  has only finitely many integral solution  $(S, T)$  for any integer  $\Delta \neq 0$ . For each of these solutions the corresponding point in  $S^3 H_C^\vee / SL(H_C)$  lies outside of the discriminant curve, so that the  $\pi$ -fiber over it is a closed  $SL(H_C)$ -orbit. The finiteness of the class number then follows from the Borel/Harish-Chandra theorem.

A famous special case of Siegel's theorem is Bachet's equation  $S^3 - T^2 = 2$ ; it has only the two obvious solutions  $(3, \pm 5)$ .

**REMARK 10.** To get finiteness results for ternary cubic forms it is not sufficient to fix the  $J$ -invariant (instead of the discriminant): The forms  $f_m = X^3 + XZ^2 + Z^3 + mY^2Z$ ,  $m \in \mathbf{Z} \setminus \{0\}$ , all have the same  $J$ -invariant, but they are not equivalent, even over  $\mathbf{Q}$ , since they have bad reduction at different primes  $p \mid m$ .

#### 4. INVARIANTS OF COMPLEX 3-FOLDS

In this section we begin to investigate the topology of 1-connected, compact, complex 3-folds. After a brief discussion of the possible systems of Chern numbers of almost complex 6-manifolds, we study the behaviour of the topological invariants of complex 3-folds under certain standard constructions, like e.g. branched coverings, or blow-ups of points and curves. Then we describe some interesting examples of 1-connected, non-Kählerian 3-folds, including a new construction method which generalizes the Calabi-Eckmann manifolds. These examples will be needed in the next section in order to realize complex types of cubic forms as cup-forms of complex 3-folds.

##### 4.1 CHERN NUMBERS OF ALMOST COMPLEX STRUCTURES

Let  $X$  be a closed, oriented, 6-dimensional differentiable manifold. The tangent bundle of  $X$  is induced by a classifying map  $t_X: X \rightarrow BSO(6)$  which is unique up to homotopy. By an almost complex structure on  $X$  we mean the homotopy class  $[\tilde{t}_X]$  of a lifting  $\tilde{t}_X: X \rightarrow BU(3)$  of  $t_X$  to  $BU(3)$ .

PROPOSITION 8. *Every closed, oriented, 6-dimensional  $C^\infty$ -manifold  $X$  without 2-torsion in  $H^3(X, \mathbf{Z})$  admits an almost complex structure. There is a 1-1 correspondence between almost complex structures on  $X$  and integral lifts  $W \in H^2(X, \mathbf{Z})$  of  $w_2(X)$ . The Chern classes  $c_i$  of the almost complex manifold  $(X, W)$  are given by  $c_1 = W$ ,  $c_2 = \frac{1}{2}(W^2 - p_1(X))$ .*

*Proof* (cf. [W]). The obstructions against lifting  $t_X$  to  $BU(3)$  lie in the cohomology groups  $H^{i+1}(X, \pi_i(SO(6)/U(3)))$ ,  $i = 0, 1, \dots, 5$ . Since  $SO(6)/U(3) = \mathbf{P}^3$  has only one nontrivial homotopy group  $\pi_2(SO(6)/U(3)) \cong \mathbf{Z}$  in dimensions  $i \leq 5$ , there is in fact only one obstruction  $o(t_X) \in H^3(X, \mathbf{Z})$ , and this obstruction can be identified with the image of  $w_2(X)$  under the Bockstein homomorphism  $\beta: H^2(X, \mathbf{Z}/_2) \rightarrow H^3(X, \mathbf{Z})$ . Since  $H^3(X, \mathbf{Z})$  has no 2-torsion by assumption,  $\beta w_2(X)$  must be equal to zero, so that  $X$  has at least one almost complex structure  $[\tilde{t}_X] \in [X, BU(3)]$ . Standard homotopy arguments show now that the map, which assigns to an almost complex structure  $[\tilde{t}_X]$  its first Chern class  $\tilde{t}_X^* c_1$ , induces a 1-1 correspondence between integral lifts  $W \in H^2(X, \mathbf{Z})$  of  $w_2(X)$  and homotopy classes of liftings of  $[t_X]$  to  $BU(3)$ .

The second Chern class  $c_2$  of the almost complex manifold  $(X, W)$  is determined by  $W^2 - 2c_2 = p_1(X)$ .

The Chern numbers  $c_1^3, c_1 c_2, c_3$  of an almost complex manifold  $X$  of real dimension 6 satisfy the following congruences:  $c_1^3 \equiv 0 \pmod{2}$ ,  $c_1 c_2 \equiv 0 \pmod{24}$ ,  $c_3 \equiv 0 \pmod{2}$ . Conversely, given a triple  $(a, b, c)$  of integers  $a \equiv 0 \pmod{2}$ ,  $b \equiv 0 \pmod{24}$ , and  $c \equiv 0 \pmod{2}$ , there always exists an almost complex manifold  $X$  of dimension 6 with Chern numbers  $c_1^3 = a$ ,  $c_1 c_2 = b$ ,  $c_3 = c$ .

It is not totally clear, however, that one can find a *connected* manifold  $X$  with prescribed Chern numbers [H1].

PROPOSITION 9. *Every triple  $(a, b, c) \in \mathbf{Z}^{\oplus 3}$  satisfying  $a \equiv 0 \pmod{2}$ ,  $b \equiv 0 \pmod{24}$ ,  $c \equiv 0 \pmod{2}$  is realizable as the Chern numbers of an almost complex 6-manifold.*

*Proof.* Consider the complete intersection  $V(f, g) \subset \mathbf{P}^5$  defined by the polynomials  $f(z) = z_0^2 + z_1^2 + 2z_2^2 - z_3^2 - z_4^2 - 2z_5^2$ , and  $g(z) = z_0^4 + z_1^4 + 2z_2^4 - z_3^4 - z_4^4 - 2z_5^4$  [We].  $V(f, g)$  is a singular 3-fold with 90 ordinary double points, and every small resolution  $V$  of these nodes is a (not necessarily projective) Calabi-Yau 3-fold with Euler number 4. Suppose now that a prescribed triple  $(a, b, c) \in \mathbf{Z}^{\oplus 3}$  is realized by a possibly disconnected almost complex manifold  $X = \coprod_{i \in I} X_i$ . If we form the connected sum

$X'$  of the  $X_i$ , we obtain a connected almost complex manifold  $X'$  with Chern numbers  $c_1^3 = a$ ,  $c_1 c_2 = b$ , but with  $c_3 = c - 2(|I| - 1)$ .

If  $|I| > 1$  take the connected sum of  $X'$  with  $|I| - 1$  copies of the complex manifold  $V$ . Since  $V$  is Calabi-Yau, the Chern numbers  $c_1^3$  and  $c_1 c_2$  remain unchanged, whereas the Euler number of  $X' \#_{|I|-1} V$  becomes  $c_3 = c$ .

REMARK 11. The above argument has been suggested by F. Hirzebruch after talk at the MPI, in which one of us had sketched a less geometric proof of the proposition.

There is another question which is related to the result above: Fix a closed, oriented, 6-dimensional differentiable manifold  $X$ . Which pairs  $(a, b)$  of integers with  $a \equiv 0 \pmod{2}$  and  $b \equiv 0 \pmod{24}$  occur as Chern numbers  $c_1^3$  and  $c_1 c_2$  of almost complex structures on  $X$ , and in how many ways?

For manifolds with  $b_2(X) = 1$  the Chern numbers determine the almost complex structure. For manifolds with  $b_2 > 1$  this is no longer true. It is possible to construct infinitely many distinct almost complex structures with the same Chern numbers on a hypersurface of bidegree  $(3, 3)$  in  $\mathbf{P}^2 \times \mathbf{P}^2$ .

An almost complex structure  $[\tilde{t}_X]$  on a differentiable 6-manifold  $X$  is said to be integrable if  $\tilde{t}_X$  is homotopic to the classifying map of a complex 3-fold. We are not aware of any example of an almost complex 6-manifold which is known not to be integrable. On the other hand, it is also unknown whether or not the Chern numbers  $c_1^3, c_1 c_2$  of integrable almost complex manifold are topological invariants. The following remark might therefore be of some interest:

PROPOSITION 10. *If the Chern numbers of complex 3-folds are topological invariants, then there exist almost complex structures which are not integrable.*

*Proof.* Consider a closed, oriented differentiable 6-manifold  $X$  without 2-torsion in  $H^3(X, \mathbf{Z})$ . Fix any almost complex structure on  $X$  with first Chern class  $W \in H^2(X, \mathbf{Z})$ .

Every element  $x \in H^2(X, \mathbf{Z})$  defines a new almost complex structure on  $X$  with first Chern class  $W + 2x$ , and it is easy to see that these two almost complex structures have the same Chern numbers if and only if  $x$  satisfies the equations  $p_1(X) \cdot x = 0$ , and  $3W^2 \cdot x + 6W \cdot x^2 + 4x^3 = 0$ .

Suppose now  $(X, W)$  is integrable,  $p_1(X) \neq 0$ , and choose  $x \in H^2(X, \mathbf{Z})$  such that  $p_1(X) \cdot x \neq 0$ . Then clearly, either none of the almost complex manifolds  $(X, W + 2x)$  is integrable, or the Chern numbers of complex 3-folds are not topologically invariant.

REMARK 12. It is very likely that there exist non-integrable almost complex structures on manifolds  $X$  as above, but probably this is hard to prove. It is also not unlikely that the Chern numbers of complex 3-folds are not topological invariants. A possible way to check this would be, to run a computer search for 3-folds given by certain standard constructions.

## 4.2 STANDARD CONSTRUCTIONS

For later use we investigate the topological invariants of complex 3-folds which can be obtained by certain simple standard constructions like complete intersections, simple cyclic coverings, blow-ups of points and curves, and projective bundles.

PROPOSITION 11 (Libgober/Wood). *Let  $X \subset \mathbf{P}^{3+r}$  be a smooth complete intersection of multidegree  $\underline{d} = (d_1, \dots, d_r)$ . Choose a normalized basis  $e \in H^2(X, \mathbf{Z})$ , and let  $\varepsilon \in H^4(X, \mathbf{Z})$  be defined by  $\varepsilon(e) = 1$ . Then the invariants of  $X$  are:*

$$F_X(xe) = dx^3 \quad \text{where } d = \prod_{i=1}^r d_i, \quad w_2(X) \equiv (4 + r - \sum_{i=1}^r d_i)e,$$

$$p_1(X) = d(4 + r - \sum_{i=1}^r d_i^2)\varepsilon, \quad \text{and}$$

$$b_3(X) = 4 - \frac{d}{6} [(4 + r - \sum_{i=1}^r d_i)^3 - 3(4 + r - \sum_{i=1}^r d_i)(4 + r - \sum_{i=1}^r d_i^2) + 2(4 + r - \sum_{i=1}^r d_i^3)].$$

*Proof.* [L/W].

PROPOSITION 12. *Let  $X$  be a smooth, 1-connected, complex projective 3-fold, and let  $\pi: X' \rightarrow X$  be a simple cyclic covering of degree  $d$  branched along a non-singular ample divisor  $B \in |L^{\otimes d}|$ .  $X'$  is smooth, projective, 1-connected, and  $\pi^*: H^2(X, \mathbf{Z}) \rightarrow H^2(X', \mathbf{Z})$  is an isomorphism. The invariants of  $X$  and  $X'$  are related by the formulae:*

$$(\pi^*)^* F_{X'} = dF_X, \quad w_2(X') - \pi^* w_2(X) \equiv (d-1)\pi^* c_1(L),$$

$$p_1(X') - \pi^* p_1(X) = (1-d)(1+d)\pi^* c_1(L)^2, \quad \text{and}$$

$$b_3(X') = db_3(X) + (d-1)(b_2(B) - 2b_2(X)).$$

*Proof.*  $X'$  is clearly smooth and projective. By a theorem of M. Cornalba  $\pi: X' \rightarrow X$  is a 3-equivalence, i.e.  $\pi_*: \pi_i(X') \rightarrow \pi_i(X)$  is bijective for  $i \leq 2$ , and surjective for  $i = 3$  [Co].  $X'$  is therefore 1-connected, and  $\pi^*: H^2(X, \mathbf{Z}) \rightarrow H^2(X', \mathbf{Z})$  is an isomorphism. The relation between  $F_{X'}$  and  $F_X$  is obvious, whereas the formula for  $b_3(X')$  follows from  $\pi_1(B) = \{1\}$  and standard properties of Euler numbers.

In order to calculate  $w_2(X')$  and  $p_1(X')$  we compute the Chern classes of  $X'$ :  $c_1(X') - \pi^*c_1(X) = (1 - d)\pi^*c_1(L)$ ,  $c_2(X') - \pi^*c_2(X) = (1 - d)\pi^*[c_1(X)c_1(L) - dc_1(L)^2]$ .

The latter formulae follow from the description of  $X'$  as a divisor in the total space of the line bundle  $L$ .

EXAMPLE 9. Let  $X$  be a  $d$ -fold, simple cyclic covering of  $\mathbf{P}^3$  branched along a smooth surface  $B \subset \mathbf{P}^3$  of degree  $dl, l \geq 1$ . Let  $e \in H^2(X, \mathbf{Z})$  correspond to the preimage of a plane in  $\mathbf{P}^3$ . The invariants of  $X$  are then given by:

$$F_X(xe) = dx^3, w_2(X) \equiv (4 + (1 - d)l)e, p_1(X) = d[4 + (1 - d)(1 + d)l^2]\varepsilon$$

$$(\varepsilon(e) = 1), b_3(X) = (d - 1)(d^2l^2 - 4dl + 6)dl.$$

PROPOSITION 13. Let  $\sigma: \hat{X} \rightarrow X$  be the blow-up of a complex 3-fold  $X$  in a point, and let  $e \in H^2(\hat{X}, \mathbf{Z})$  be the class of the exceptional divisor. The invariants of  $\hat{X}$  and  $X$  are related by the following formulae:

$$F_{\hat{X}}(\sigma^*h + xe) = F_X(h) + x^3 \quad \forall h \in H^2(X, \mathbf{Z}), x \in \mathbf{Z}, w_2(\hat{X}) = \sigma^*w_2(X),$$

$$p_1(\hat{X}) = \sigma^*p_1(X) + 4(e^2 - \sigma^*c_1(X) \cdot e), b_3(\hat{X}) = b_3(X).$$

*Proof.* Standard arguments, see [G/H]. The Chern classes are related by  $c_1(\hat{X}) = \sigma^*c_1(X) - 2e, c_2(\hat{X}) = \sigma^*c_2(X)$ .

PROPOSITION 14. Let  $\sigma: \hat{X} \rightarrow X$  be the blow-up of a complex 3-fold  $X$  along a smooth curve  $C$  of genus  $g$ , and let  $e \in H^2(\hat{X}, \mathbf{Z})$  be the class of the exceptional divisor. The invariants of  $\hat{X}$  and  $X$  are related by:

$$F_{\hat{X}}(\sigma^*h + xe) = F_X(h) - 3h \cdot Cx^2 - \text{deg}N_{C/X}x^3 \quad \forall h \in H^2(X, \mathbf{Z}),$$

$$x \in \mathbf{Z}, w_2(\hat{X}) \equiv \sigma^*w_2(X) + e, p_1(\hat{X}) = \sigma^*p_1(X) + (e^2 - 2\sigma^*C),$$

$$b_3(\hat{X}) = b_3(X) + 2g.$$

*Proof.* [G/H]. The Chern classes are given by  $c_1(\hat{X}) = \sigma^*c_1(X) - c, c_2(\hat{X}) = \sigma^*(c_2(X) + C) - \sigma^*c_1(X) \cdot e$ .

PROPOSITION 15. Let  $E$  be a holomorphic vector bundle of rank 2 with Chern classes  $c_i(E), i = 1, 2$  over a 1-connected, compact complex surface  $Y$ , and let  $\pi: \mathbf{P}(E) \rightarrow Y$  be the projective bundle of lines in the fibers of  $E$ . The cup-form of  $\mathbf{P}(E)$  is given by

$$F_{\mathbf{P}(E)}(h + x\xi) = x[(3h^2) - (3c_1(E) \cdot h)x + (c_1(E)^2 - c_2(E))x^2],$$

where  $\xi = c_1(\mathcal{O}_{\mathbf{P}(E)}(1))$ ,  $h \in H^2(Y, \mathbf{Z})$ , and  $x \in \mathbf{Z}$ . The other topological invariants of  $\mathbf{P}(E)$  are:

$$\begin{aligned} w_2(\mathbf{P}(E)) &\equiv \pi^*(w_2(Y) + c_1(E)), p_1(E)) \\ &= \pi^*[c_1(Y)^2 - 2c_2(Y) + c_1(E)^2 - 4c_2(E)], b_3(\mathbf{P}(E)) = 0. \end{aligned}$$

*Proof.* The Leray-Hirsch theorem identifies the cohomology ring  $H^*(\mathbf{P}(E), \mathbf{Z})$  with the ring  $H^*(Y, \mathbf{Z})[\xi]/\langle \xi^2 + c_1(E) \cdot \xi + c_2(E) \rangle$ ; this determines the cup-form. In order to calculate the characteristic classes one uses the exact sequence  $0 \rightarrow \mathcal{O}_{\mathbf{P}(E)} \rightarrow \pi^*E \otimes \mathcal{O}_{\mathbf{P}(E)}(1) \rightarrow T_{\mathbf{P}(E)} \rightarrow \pi^*T_Y \rightarrow 0$ .  $b_3(\mathbf{P}(E)) = 0$  follows from  $b_1(Y) = 0$  and the Leray-Hirsch theorem.

### 4.3 EXAMPLES OF 1-CONNECTED NON-KÄHLERIAN 3-FOLDS

Recall that the Hessian of a symmetric trilinear form  $F \in S^3H^\vee$  on a free  $\mathbf{Z}$ -module  $H$  of finite rank was defined as the composition  $H_F: H \xrightarrow{F^t} S^2H^\vee \xrightarrow{\text{disc}} \mathbf{Z}$ . In terms of coordinates  $\xi_1, \dots, \xi_b$  on  $H$  it is given by the determinant  $\det\left(\frac{\partial^2 f}{\partial \xi_i \partial \xi_j}\right)$ , where  $f \in \mathbf{C}[H_{\mathbf{C}}]_3$  is the homogeneous cubic polynomial associated with  $F$ .

**PROPOSITION 16.** *Let  $F$  be a symmetric trilinear form whose Hessian vanishes identically. Then  $F$  is not realizable as cup-form of a Kählerian 3-fold.*

*Proof.* Let  $X$  be a complex 3-fold with a Kähler metric  $g$ . The Kähler class  $[\omega_g] \in H^2(X, \mathbf{R})$  defines a multiplication map  $\cdot [\omega_g]: H^2(X, \mathbf{R}) \rightarrow H^4(X, \mathbf{R})$ , which is an isomorphism by the Hard Lefschetz Theorem [G/H]. In section 3.1 we have seen that this is not possible if the Hessian of the cup-form vanishes.

**COROLLARY 6.** *Cubic forms  $f \in \mathbf{C}[H_{\mathbf{C}}]_3$  which depend on strictly less than  $b = \text{rk}_{\mathbf{Z}}H$  variables are not realizable as cup-forms of Kählerian 3-folds with  $b_2 = b$ .*

By considering the Hessian of a cup-form over the reals one obtains further conditions.

**DEFINITION 4.** *Let  $F \in S^3H^\vee$  be a symmetric trilinear form on a free  $\mathbf{Z}$ -module of rank  $b$ .*

*The Hesse cone of  $F$  is the subset  $\mathcal{H}_F \subset H_{\mathbf{R}}$  defined by  $\mathcal{H}_F := \{h \in H_{\mathbf{R}} \mid (-1)^b \det(F^t(h)) < 0\}$ .*

The index cone  $\mathcal{J}_F$  of  $F$  is the subset  $\mathcal{J}_F := \{h \in \mathcal{H}_F \mid F^t(h) \in S^2 H_{\mathbf{R}}^{\vee}\}$  has signature  $(1, -1, \dots, -1)$ .

Clearly  $\mathcal{J}_F$  is an open subcone of  $\mathcal{H}_F$  which coincides with  $\mathcal{H}_F$  iff  $b \leq 2$ .

**THEOREM 5.** *Let  $F_X \in S^3 H^2(X, \mathbf{Z})^{\vee}$  be the cup-form of a smooth projective 3-fold with  $h^{0,2}(X) = 0$ . Then  $F_X$  has a non-empty index cone.*

*Proof.* Let  $h \in H^2(X, \mathbf{Z})$  be the dual class of a hyperplane section  $Y$  in some projective embedding. The inclusion  $i: Y \hookrightarrow X$  induces a monomorphism  $i^*: H^2(X, \mathbf{Z}) \rightarrow H^2(Y, \mathbf{Z})$  by the weak Lefschetz theorem. The symmetric bilinear form  $F_X^t(h) \in S^2 H^2(X, \mathbf{Z})^{\vee}$  is simply the pull-back of the cup-form of  $Y$  under the inclusion  $i^*$ ; it is therefore non-degenerate by the Hard Lefschetz theorem [L]. Applying the Hodge index theorem to  $Y$  we see that the real bilinear form  $F_X^t(h) \in S^2 H^2(X, \mathbf{R})^{\vee}$  must have one positive and  $b - 1$  negative eigenvalues. In other words:  $h \in I_{F_X}$ .

**REMARK 13.** This result has two applications: it provides topological ‘upper bounds’ for the ample cone of a projective 3-fold with  $h^{0,2} = 0$ , and it gives further restrictions on symmetric trilinear forms to be realizable as cup-forms of projective 3-folds with  $h^{0,2} = 0$  if  $b \geq 4$ .

These applications will be discussed in section 5.

We will now describe examples of 1-connected, non-Kählerian, complex 3-folds and determine their topological structure.

**EXAMPLE 10 (Calabi-Eckmann).** E. Calabi and B. Eckmann have defined complex structures  $X_{\tau}$ , depending on a parameter  $\tau$ , on the product  $S^3 \times S^3 [C/E]$ . Their manifolds are principal fiber bundles over  $\mathbf{P}^1 \times \mathbf{P}^1$  whose fiber and structure group is the elliptic curve  $E_{\tau} = \mathbf{C}/\mathbf{Z} \oplus \mathbf{Z}\tau$ ,  $\text{Im}(\tau) > 0$ .

The Calabi-Eckmann manifolds are homogeneous, non-Kählerian 3-folds of algebraic dimension 2.

**EXAMPLE 11 (Maeda).** H. Maeda has generalized the Calabi-Eckmann construction. He constructed fiber bundles  $X'_{\tau}$  over Hirzebruch surfaces  $\mathbf{F}_n, n \geq 0$ , whose fiber and structure group are an elliptic curve  $E_{\tau}$  and  $\text{Aut}(E_{\tau})$  respectively [M].  $X'_{\tau}$  is again diffeomorphic to  $S^3 \times S^3$ , and therefore non-Kählerian. Maeda’s manifolds  $X'_{\tau}$  are homogeneous if and only if  $n = 0$  in which case they are Calabi-Eckmann 3-folds.



The Calabi-Eckmann construction can also be generalized in the following way:

Let  $S^2 \tilde{\times} S^4$  be the non-trivial  $S^4$ -bundle over  $S^2$ , i.e.  $S^2 \tilde{\times} S^4$  is the unique 1-connected, closed, oriented, differentiable 6-manifold with  $H_2(S^2 \tilde{\times} S^4, \mathbf{Z}) \cong \mathbf{Z}$  and  $b_3 = 0$ , whose cup-form and Pontrjagin class vanish, but whose Stiefel-Whitney class  $w_2$  is non-zero.

**THEOREM 6.** *For any integer  $b \geq 0$  there exist compact complex 3-folds  $X_b$ , and  $X_b^-$  if  $b \geq 1$ , which are homeomorphic to  $\#_b S^2 \times S^4 \#_{b+1} S^3 \times S^3$ , and  $S^2 \tilde{\times} S^4 \#_{b-1} S^2 \times S^4 \#_{b+1} S^3 \times S^3$ .*

*Proof.* Let  $Y$  be a 1-connected, compact complex surface with  $p_g(Y) = 0$  and  $b_2(Y) \geq 2$ , and let  $E = \mathbf{C}/\Gamma$  be the elliptic curve associated to the lattice  $\Gamma \subset \mathbf{C}$ . We want to construct the required 3-folds as total spaces of principal  $E$ -bundles over  $Y$ . Let  $c: H_2(Y, \mathbf{Z}) \rightarrow \Gamma$  be an arbitrary epimorphism. The corresponding cohomology class  $c \in H^2(Y, \Gamma)$  defines a topological principal bundle over  $Y$  with fiber and structure group  $E = \mathbf{C}/\Gamma$  as follows immediately from the identification of the classifying space  $BE \simeq K(\Gamma, 2)$ .

Let  $\mathcal{O}_Y(E)$  be the sheaf of germs of holomorphic maps from  $Y$  to  $E$ . We have a short exact sequence  $0 \rightarrow \Gamma \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_Y(E) \rightarrow 0$  and a corresponding exact cohomology sequence

$$\rightarrow H^1(Y, \mathcal{O}_Y) \rightarrow H^1(Y, \mathcal{O}_Y(E)) \xrightarrow{\delta} H^2(Y, \Gamma) \rightarrow H^2(Y, \mathcal{O}_Y) \rightarrow$$

By our assumptions  $\delta$  is an isomorphism, so that every topological principal  $E$ -bundle admits a holomorphic structure. Let  $X$  be the total space of such a bundle corresponding to a surjective map  $c: H_2(Y, \mathbf{Z}) \rightarrow \Gamma$ . The homotopy sequence of the fibration  $p: X \rightarrow Y$  yields the sequence

$$0 \rightarrow \pi_2(X) \xrightarrow{p^*} \pi_2(Y) \rightarrow \pi_1(E) \rightarrow \pi_1(X) \xrightarrow{p^*} \pi_1(Y) \rightarrow 0.$$

Since  $Y$  is 1-connected,  $\pi_2(Y)$  can be identified with  $H_2(Y, \mathbf{Z})$ , and then the boundary map  $\pi_2(Y) \rightarrow \pi_1(E)$  becomes the characteristic map  $c: H_2(Y, \mathbf{Z}) \rightarrow \Gamma$  of the bundle. This implies  $\pi_1(X) = \{1\}$ , whereas  $H_2(X, \mathbf{Z})$  is given by:  $0 \rightarrow H_2(X, \mathbf{Z}) \xrightarrow{p^*} H_2(Y, \mathbf{Z}) \xrightarrow{c} \Gamma \rightarrow 0$ .

In particular,  $H_2(X, \mathbf{Z})$  is free as a submodule of  $H_2(Y, \mathbf{Z})$ , and by dualizing the last sequence we obtain an identification (via  $p^*$ )

$$H^2(X, \mathbf{Z}) = H^2(Y, \mathbf{Z})/\Gamma^\vee.$$

The cup-form  $F_X$  of  $X$  is therefore trivial. In order to calculate  $p_1(X)$  and  $w_2(X)$ , we use the exact sequence of tangent sheaves:  $0 \rightarrow T_{X/Y} \rightarrow T_X$

$\rightarrow p^*T_Y \rightarrow 0$ . Since  $T_{X/Y}$  is a trivial bundle, the characteristic classes of  $X$  are simply the pullbacks of the corresponding classes of  $Y$ . But the map  $p^*: H^4(Y, \mathbf{Z}) \rightarrow H^4(X, \mathbf{Z})$  is zero, since  $\langle p^*(\varepsilon) \cup p^*(\alpha), [X] \rangle = \langle \varepsilon \cup \alpha, p_*[X] \rangle = 0$  for all classes  $\varepsilon \in H^4(Y, \mathbf{Z})$ , and  $\alpha \in H^2(Y, \mathbf{Z})$ .

Thus  $p_1(X) = 0$ , and  $w_2(X)$  is the residue of  $w_2(Y) \in H^2(Y, \mathbf{Z}/_2)$  modulo  $\Gamma^\vee/_{2\Gamma^\vee}$ .

The Euler characteristic of  $X$  is zero, so that from  $b_2(X) = b_2(Y) - 2$  we find  $b_3(X) = 2(b_2(Y) - 1)$ . The system of invariants associated to the manifold  $X$  is therefore given by

$$(b_2(Y) - 1, H^2(Y, \mathbf{Z})/\Gamma^\vee, w_2(Y) \pmod{\Gamma^\vee/_{2\Gamma^\vee}}, 0, 0, 0),$$

i.e.  $X$  is diffeomorphic to

$$\#_{b_2(Y)-2} S^2 \times S^4 \#_{b_2(Y)-1} S^3 \times S^3 \text{ if } w_2(Y) \in \Gamma^\vee/_{2\Gamma^\vee},$$

and to  $S^2 \tilde{\times} S^4 \#_{b_2(Y)-3} S^2 \times S^4 \#_{b_2(Y)-1} S^3 \times S^3$  if  $b_2(Y) \geq 3$ , and  $w_2(Y) \notin \Gamma^\vee/_{2\Gamma^\vee}$ .

EXAMPLE 12 (Kato). In the two papers [K1], [K2] M. Kato studies the class of compact, complex 3-folds  $X$  containing smooth rational curves with neighborhoods biholomorphic to those of projective lines in  $\mathbf{P}^3$ . On this class of 3-folds, called class  $L$ , he defines a semi-group structure  $+$  with neutral element  $\mathbf{P}^3$ .

Kato's connecting operation  $+$  is defined by removing 'lines'  $L_i \subset X_i$  from 3-folds  $X_i, i = 1, 2$ , and by identifying the complements  $X_i \setminus L_i$  along open sets  $U_i \setminus L_i$  obtained from suitable neighborhoods  $U_i \subset X_i$ .

Starting with a certain elliptic fiber space  $X_1$  over the blow-up of  $\mathbf{P}^1 \times \mathbf{P}^1$  in a point, he constructs a sequence of 3-folds  $X_n := X_1 + X_{n-1}, n \geq 2$ . The 3-folds  $X_n$  are 1-connected spin-manifolds with  $H_2(X_n, \mathbf{Z}) = \mathbf{Z}$ . Their cup-forms  $F_{X_n}$ , and their Pontrjagin classes  $p_1(X_n)$  are in terms of a (normalized) generator  $e_n \in H^2(X_n, \mathbf{Z})$  and its dual class  $\varepsilon_n \in H^4(X_n, \mathbf{Z})$  given by  $F_{X_n}(xe_n) = (n-1)x^3$ , and  $p_1(X_n) = 4(n-1)\varepsilon_n$  ( $\varepsilon_n(e_n) = 1$ ). The third Betti-number of  $X_n$  is  $4n$ .

In particular,  $X_1$  is diffeomorphic to  $S^2 \times S^4 \#_2 S^3 \times S^3$ , and  $X_2$  is diffeomorphic to  $\mathbf{P}^3 \#_4 S^3 \times S^3$ . It is interesting to note that the Chern-numbers  $c_1^3, c_1 c_2$  of the  $X_n$  are  $c_1^3 = 64(1-n), c_1 c_2 = 24(1-n)$ , i.e. they satisfy  $8c_1 c_2 = 3c_1^3$ . For projective manifolds of general type this equality is characteristic for ball quotients [Y].

EXAMPLE 13 (Twistor spaces). Let  $p: Z \rightarrow M$  be the twistor fibration of a closed, oriented Riemannian 4-manifold  $(M, g)$ .  $Z$  carries a natural almost complex structure which is integrable if and only if  $g$  is self-dual [A/H/S].

Examples of 1-connected 4-manifolds which admit self-dual structures are  $S^4$ ,  $\#_n \mathbf{P}^2$ , and  $K3$ -surfaces.

The total spaces of their twistor fibrations are 1-connected complex 3-folds which may be Moishezon for  $S^4$  and  $\#_n \mathbf{P}^2$  [C], but which are usually non-Kähler [Hi]. We leave it to the reader to calculate the topological invariants of these 3-folds. There is an interesting relation between Twistor spaces of connected sums and Kato's connection operation  $+$  for class  $L$  manifolds [K2], [D/F].

EXAMPLE 14 (Oguiso). In a recent preprint [O1] K. Oguiso constructs examples of 1-connected, Moishezon Calabi-Yau 3-folds with very interesting cup-forms. He proves that for every integer  $d \geq 1$  there exists a smooth complete intersection  $X'_d$  of type  $(2, 4)$  in  $\mathbf{P}^5$  which contains a non-singular rational curve  $C_d$  of degree  $d$  with normal bundle  $N_{C_d/X'_d} = \mathcal{O}_{C_d}(-1)^{\oplus 2}$ .

The 3-fold  $X'_d$  can now be flopped along  $C_d$ , i.e.  $C_d$  can be blown up to  $\mathbf{P}(N_{C_d/X'_d}) \cong \mathbf{P}^1 \times \mathbf{P}^1$ , and then 'blown down in the other direction'. The resulting 3-fold  $X_d$  is a 1-connected Moishezon manifold with trivial canonical bundle and cup-form  $F_{X_d}$  given by  $F_{X_d}(xe_d) = (d^3 - 8)x^3$ . Here  $e_d \in H^2(X_d, \mathbf{Z})$  is the normalized generator corresponding to the strict transform of the negative of a hyperplane section of  $X'_d$ . The Pontrjagin class of  $X_d$  is  $p_1(X_d) = (112 + 4d)\varepsilon_d$  where  $\varepsilon_d \in H^4(X_d, \mathbf{Z})$  denotes the generator with  $\varepsilon_d(e_d) = 1$ . Since the Euler-number does not change under a flop we have  $b_3(X_d) = 180$  for every  $d$ .

## 5. COMPLEX 3-FOLDS WITH SMALL $b_2$

In this section we investigate the following natural problem: Which cubic forms can be realized as cup-forms of compact complex 3-folds? For small  $b_2$  something can be said: Any core of a 1-connected, closed, oriented differentiable 6-manifold with  $H_2(X, \mathbf{Z}) \cong \mathbf{Z}$  is homotopy equivalent to the core of a 1-connected complex 3-fold. In the case  $b_2 = 2$ , at least every discriminant  $\Delta$  is realizable by a complex manifold. If  $b_2 = 3$  we can realize all types of complex cubics with one exception, the union of a smooth conic and a tangent line. In addition to these realization results we prove a finiteness