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REMARK 6. It is not difficult to show that  $F^t$  is not injective if and only if there exists a proper quotient  $\bar{H}_C$  of  $H_C$ , and a form  $\bar{F} \in S^3 \bar{H}_C^\vee$  whose pull-back to  $H_C$  is the given form  $F$ . This means that the Hessians of cubic polynomials  $f \in \mathbf{C}[H_C]_3$  which ‘do not depend on all variables’ are automatically zero.

The converse holds for forms in  $b \leq 4$  variables, but not in general [G/N].

### 3.2 THE GIT QUOTIENT $S^3 H_C^\vee //_{SL(H_C)}$

Let  $V := S^3 H_C^\vee$  be the vector space of complex cubic forms. The reductive group  $G := SL(H_C)$  acts rationally on  $V$ , and therefore has a finitely generated ring  $\mathbf{C}[V]^G$  of invariants [H]. The inclusion  $\mathbf{C}[V]^G \subset \mathbf{C}[V]$  induces a regular map  $\pi: V \rightarrow V//_G$  onto the affine variety  $V//_G$  with coordinate ring  $\mathbf{C}[V]^G$ . It is well known that  $\pi$  is a categorical quotient, which is  $G$ -closed and  $G$ -separating, so that  $V//_G$  parametrizes precisely the closed  $G$ -orbits in  $V$ . Recall that a point  $v \in V$  is semi-stable if  $0 \notin \bar{G} \cdot v$ , and that  $v$  is stable if  $G \cdot v$  is closed in  $V$  and the isotropy group  $G_v$  is finite [M/F]. Denote the  $G$ -invariant, open subsets of semistable (stable) points in  $V$  by  $V^{ss}$  ( $V^s$ ).

The complement  $V \setminus V^{ss} = \pi^{-1}(\pi(0))$  consists of ‘Nullformen’, i.e. forms for which all polynomial invariants vanish. The open subset of stable points, which includes in particular all non-singular forms, has a geometric quotient, given by the restricted map  $\pi|_{V^s}: V^s \rightarrow \pi(V^s)$ .

REMARK 7. Let  $A_\circ \in GL(H)$  be a fixed automorphism of determinant  $\det A_\circ = -1$ , e.g.  $A_\circ = -id_H$  if  $b$  is odd.  $A_\circ$  induces a  $\mathbf{Z}/2$ -action on  $S^3 H^\vee /_{SL(H)}$  and on  $S^3 H_C^\vee /_{SL(H_C)}$ , for which the map  $c$  is equivariant.

Let  $\hat{G} \subset GL(H_C)$  be the semi-direct product of  $SL(H_C)$  and  $\mathbf{Z}/2$  generated by  $A_\circ$  and  $SL(H_C)$ . The invariant ring  $\mathbf{C}[V]^{\hat{G}}$  has an important topological interpretation: it consists of all polynomial invariants of complex cohomology rings of 1-connected, closed, oriented 6-dimensional manifolds with torsion-free homology.

#### EXAMPLE 5. Binary cubics ( $b = 2$ )

Choose linear coordinates  $X, Y$  on  $H_C$ , and write a cubic polynomial  $f \in \mathbf{C}[X, Y]_3$  in the form  $f = a_0 X^3 + 3a_1 X^2 Y + 3a_2 X Y^2 + a_3 Y^3$ .

We use  $a_0, a_1, a_2, a_3$  as coordinates on  $S^3 H_C^\vee$ , so that  $\mathbf{C}[S^3 H_C^\vee] = \mathbf{C}[a_0, a_1, a_2, a_3]$ . The discriminant  $\Delta(f)$  of  $f$  is a homogeneous

polynomial of degree 4 in the coefficients  $a_0, a_1, a_2, a_3$ , explicitly given by  $\Delta(f) = a_0^2 a_3^2 - 3a_1^2 a_2^2 - 6a_0 a_1 a_2 a_3 + 4a_0 a_2^3 + 4a_1^3 a_3$ .

The discriminant generates the ring of  $SL(H_C)$ -invariants,

$$\mathbf{C}[S^3 H_C^\vee]^{SL(H_C)} = \mathbf{C}[\Delta],$$

and it is easy to see that  $\Delta$  is also  $\mathbf{Z}/2$ -invariant. A cubic form  $f$  is stable if and only if it is semistable, if and only if it is non-singular [N]. The cone of nullforms  $\pi^{-1}(\pi(0))$  is the affine hypersurface  $(\Delta)_\circ \subset S^3 H_C^\vee$ ; it has a nice geometric interpretation in terms of the Hessian. The Hessian of the cubic  $f$  is the quadratic form

$$H_f = 6^2 [(a_0 a_2 - a_1^2) X^2 + (a_0 a_3 - a_1 a_2) XY + (a_1 a_3 - a_2^2) Y^2].$$

The set of forms  $f$  with vanishing Hessians  $H_f$  form the affine cone over the rational normal curve in  $\mathbf{P}(S^3 H_C^\vee)$ ; the hypersurface of nullforms is the cone over the tangential scroll of this curve. There are 4 different types of  $SL(H_C)$ -orbits in  $S^3 H_C^\vee$ , represented by the normal forms  $XY(X + \lambda Y)$ ,  $X^2 Y$ ,  $X^3$ ,  $0$ . The first type is stable, the others are nullforms, the orbits of  $X^3$  and  $0$  have vanishing Hessians.

#### EXAMPLE 6. Ternary cubics ( $b = 3$ )

The ring of  $SL(H_C)$ -invariants of ternary cubics is a weighted polynomial ring in 2 variables,  $\mathbf{C}[S^3 H_C^\vee]^{SL(H_C)} = \mathbf{C}[S, T]$  whose generators  $S, T$  have been found by S. Aronhold [A].  $S$  is a homogeneous polynomial of degree 4 in the coefficients of a cubic  $f$ ,  $T$  is homogeneous of degree 6, both polynomials are  $\mathbf{Z}/2$ -invariant. For a cubic of the form  $f = aX^3 + bY^3 + cZ^3 + 6dXYZ$ ,  $S$  and  $T$  are given by  $S = 4d(d^3 - abc)$  and  $T = 8d^6 + 20abc(d^3 - abc)$  respectively [P]. The general formulae, which take two pages to write down, can be found in the book of Sturmfels [St]. The discriminant of a form  $f$  is homogeneous of degree 12 in the coefficients of  $f$ ; in terms of Aronhold's invariants  $S, T$  it is simply given by  $\Delta = S^3 - T^2$ . We obtain the following overall picture: The GIT quotient for ternary cubics is an affine plane  $\mathbf{A}^2$  with coordinates  $S, T$ . The complement  $\mathbf{A}^2 \setminus (\Delta)_\circ$  of the discriminant curve is the geometric quotient of stable cubics. The  $\pi$ -fibers over a point  $(S, T) \neq (0, 0)$  on the discriminant curve  $(\Delta)_\circ$  consist of 3 types of  $SL(H_C)$ -orbits: nodal cubics with normal form  $X^3 + Y^3 + 6\alpha XYZ$ , reducible cubics formed by a smooth conic and a transversal line (normal form:  $X^3 + 6\alpha XYZ$ ), and cubics consisting of three lines in general position (normal form:  $6\alpha XYZ$ ); these cubics are properly

semi-stable for  $\alpha \neq 0$  with Aronhold invariants  $S = 4\alpha^4$ ,  $T = 8\alpha^6$ . The fiber of  $\pi$  over 0 contains 6 orbits with normal forms

$$Y^2Z - X^3, Y(X^2 - YZ), XY(X + Y), X^2Y, X^3,$$

and 0, of which the last 4 types have vanishing Hessians. For more details we refer to H. Kraft's book [Kr].

REMARK 8. The natural  $\mathbf{C}^*$ -action  $f \rightarrow \lambda \cdot f$  on cubic forms induces a weighted action on the GIT quotient  $S^3 H_{\mathbf{C}}^{\vee} / SL(H_{\mathbf{C}})$ ,  $\lambda \cdot (S, T) = (\lambda^4 S, \lambda^6 T)$ . The associated weighted projective space  $\mathbf{P}^1(4, 6)$  with homogeneous coordinates  $\langle S, T \rangle$  is the good quotient for semi-stable plane cubic curves. Its affine part  $\mathbf{P}^1 \setminus (\Delta)_{\circ}$  is the moduli space of genus-1 curves. The  $PGL(H_{\mathbf{C}})$ -invariant  $J := \frac{S^3}{\Delta}$  gives the  $J$ -invariant of the corresponding curve.

### 3.3 ARITHMETICAL ASPECTS

Let  $c: S^3 H^{\vee} / SL(H) \rightarrow S^3 H_{\mathbf{C}}^{\vee} / SL(H)$  be the map which associates to the  $SL(H)$ -orbit  $\langle F \rangle$  of a symmetric trilinear form  $F \in S^3 H^{\vee}$  the  $SL(H_{\mathbf{C}})$ -orbit  $\langle F \rangle_{\mathbf{C}}$  of its complexification. The  $c$ -fiber over  $\langle F \rangle_{\mathbf{C}}$  can be identified with the subset  $(SL(H_{\mathbf{C}}) \cdot F \cap S^3 H^{\vee}) / SL(H)$  of  $S^3 H^{\vee} / SL(H)$ . C. Jordan has shown that these subsets are finite provided the cubic form  $f \in \mathbf{C}[H_{\mathbf{C}}]_3$  associated to  $F$  has a non-vanishing discriminant [J1]. Jordan's original proof, which is only two pages long, is somewhat hard to follow. The following theorem of A. Borel and Harish-Chandra provides, however, a vast generalization of Jordan's finiteness result:

THEOREM 3 (Borel/Harish-Chandra). *Let  $G$  be a reductive  $\mathbf{Q}$ -group,  $\Gamma \subset G$  an arithmetic subgroup,  $\xi: G \rightarrow GL(V)$  a  $\mathbf{Q}$ -morphism, and  $L \subset V$  a  $\Gamma$ -invariant sublattice of  $V_{\mathbf{Q}}$ . If  $v \in V$  has a closed  $G$ -orbit in  $V$ , then  $G_v \cap L/\Gamma$  is a finite set.*

*Proof.* [B].

COROLLARY 4. *Let  $F \in S^3 H^{\vee}$  be a symmetric trilinear form on  $H$ . If the  $SL(H_{\mathbf{C}})$ -orbit of  $F$  in  $S^3 H_{\mathbf{C}}^{\vee}$  is closed, then the fiber  $c^{-1}(\langle F \rangle_{\mathbf{C}})$  over  $\langle F \rangle_{\mathbf{C}}$  is finite.*

To check whether a  $SL(H_{\mathbf{C}})$ -orbit  $SL(H_{\mathbf{C}}) \cdot F$  is closed in  $S^3 H_{\mathbf{C}}^{\vee}$ , one has a generalization of the Hilbert-criterion [Kr]:  $SL(H_{\mathbf{C}}) \cdot F$  is closed in  $S^3 H_{\mathbf{C}}^{\vee}$  if and only if for every 1-parameter subgroup  $\lambda: \mathbf{C}^* \rightarrow SL(H_{\mathbf{C}})$ , for