

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 41 (1995)  
**Heft:** 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** CUBIC FORMS AND COMPLEX 3-FOLDS  
**Autor:** Okonek, Ch. / Van de Ven, A.  
**Kapitel:** 3.1 Algebraic properties of cubic forms  
**DOI:** <https://doi.org/10.5169/seals-61829>

#### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

#### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

#### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 23.01.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

### 3. ALGEBRA AND ARITHMETIC OF CUBIC FORMS

Let  $H$  be a finitely generated free  $\mathbf{Z}$ -module of rank  $b$ . In this section we want to study algebraic and arithmetic properties of symmetric trilinear forms  $F \in S^3 H^\vee$  on  $H$  which admit characteristic elements; ultimately we would like to describe the classification of those forms under the action of the general linear group  $GL(H)$ , i.e. we like to investigate (part of) the quotient  $S^3 H^\vee /_{GL(H)}$ .

From what we have said in sections 1 and 2, this is clearly equivalent to classifying the cohomology rings of 1-connected, closed, oriented, 6-dimensional manifolds without torsion, and with  $b_2 = b$ ,  $b_3 = 0$ . Furthermore, up to finite indeterminacy, this is also equivalent to classifying the homotopy types of these manifolds.

The proper setting for this arithmetic moduli problem can be found in C. Seshadri's paper [S]; here we investigate only its set-theoretic aspects. Let  $H_C := H \otimes_{\mathbf{Z}} \mathbf{C}$  be the complexification of  $H$ , and let  $S^3 H_C^\vee /_{SL(H_C)}$  be the quotient of the reductive group  $SL(H_C)$ . We obtain a natural map  $c: S^3 H^\vee /_{SL(H)} \rightarrow S^3 H_C^\vee /_{SL(H_C)}$ , which allows us to break up the problem into three parts: the description of the quotient  $S^3 H_C^\vee /_{SL(H_C)}$ , the investigation of the fibers of  $c$ , and the study of the remaining  $\mathbf{Z}_2$ -action on  $S^3 H^\vee /_{SL(H)}$  which is induced by the choice of an arbitrary automorphism  $A_\circ \in GL(H)$  of determinant  $\det A_\circ = -1$ .

#### 3.1 ALGEBRAIC PROPERTIES OF CUBIC FORMS

Let  $H_C = H \otimes_{\mathbf{Z}} \mathbf{C}$  be as above, and denote by  $\mathbf{C}[H_C]_3$  the space of homogeneous polynomials of degree 3 on  $H_C$ . There exists a linear polarization operator  $\text{Pol}: \mathbf{C}[H_C]_3 \rightarrow S^3 H_C^\vee$ , sending a homogeneous cubic polynomial  $f \in \mathbf{C}[H_C]_3$  to the symmetric trilinear form  $F = \text{Pol}(f) \in S^3 H_C^\vee$  which is related to  $f$  by the identity  $F(h, h, h) = 6f(h)$ . We will usually not distinguish between a cubic polynomial  $f$  and its associated form  $F = \text{Pol}(f)$ . On  $S^3 H_C^\vee$  there exists a polynomial function  $\Delta: S^3 H_C^\vee \rightarrow \mathbf{C}$ , the discriminant, which is homogeneous of degree  $b \cdot 2^{b-1}$ , and vanishes in a form  $F$  if and only if the associated cubic hypersurface  $(f)_\circ \subset \mathbf{P}(H_C)$  has a singular point;  $\Delta$  is defined over  $\mathbf{Z}$  and is clearly invariant under the natural action of  $SL(H_C)$ .

REMARK 4. Of course, a discriminant function  $\Delta$  exists for forms of arbitrary degree  $d$ ; in the general case  $\Delta$  is homogeneous of degree  $b \cdot (d-1)^{b-1}$  on  $S^d H_C^\vee$ .

**PROPOSITION 5.** *Fix a symmetric trilinear form  $F \in S^3 H_C^\vee$  and an element  $h \in H_C \setminus \{0\}$  with  $f(h) = 0$ . The associated point  $\langle h \rangle \in \mathbf{P}(H_C)$  is a singular point of the cubic hypersurface  $(f)_0 \subset \mathbf{P}(H_C)$  if and only if the linear form  $h^2 \in H_C^\vee$  is zero. The existence of at least one such point is equivalent to the vanishing of the discriminant.*

*Proof.* From  $f(h + tv) = f(h) + 3th^2 \cdot v + 3t^2h \cdot v^2 + t^3v^3$  for every  $v \in H_C$ ,  $t \in \mathbf{C}$  we find  $\frac{d}{dt}|_0 f(h + tv) = 3h^2 \cdot v$ , i.e.  $h^2 \in H_C^\vee$  defines the differential of  $f$  in  $h$ .

**REMARK 5.**  $\mathbf{Q}$ -rational points in  $(f)_0 \subset \mathbf{P}(H_C)$ , and  $\mathbf{Q}$ -rational singularities of  $(f)_0$  have geometric significance if the cubic  $f$  is defined by the cup-form of a 6-manifold  $X$ . In fact, integral classes  $h \in H^2(X, \mathbf{Z})$  correspond to homotopy classes of maps to  $\mathbf{P}_C^3$ ; such a map factors over  $\mathbf{P}_C^2 \subset \mathbf{P}_C^3$  if and only if  $h^3 = 0$ ; if it factors over  $\mathbf{P}_C^1 \subset \mathbf{P}_C^3$ , then clearly  $h^2 = 0$ . The converse will probably not always be true since, in general, the cohomology ring does not determine the homotopy type.

In addition to the invariant discriminant  $\Delta(f)$  of a polynomial  $f$ , we will also need a fundamental covariant  $H_f$ , the Hessian of  $f$ . Let  $F = \text{Pol}(f) \in S^3 H_C^\vee$  be the polarization of  $f \in \mathbf{C}[H_C]_3$ ; the Hessian of  $f$  can then be defined as the composition  $H_f: H_C \xrightarrow{F^t} S^2 H_C^\vee \xrightarrow{\text{disc}} \mathbf{C}$ , i.e.  $H_f$  is the homogeneous polynomial function of degree  $b$  on  $H_C$  given by  $H_f(h) = \text{disc}(F^t(h))$ . In terms of linear coordinates  $\xi_1, \dots, \xi_b$  on  $H$  one finds the more familiar expression  $H_f = \det\left(\frac{\partial^2}{\partial \xi_i \partial \xi_j} f\right)$ .

**PROPOSITION 6.** *Let  $F \in S^3 H_C^\vee$  be a symmetric trilinear form. The Hessian of  $F$  is identically zero if and only if there exists no element  $h \in H_C$  for which the map  $\cdot h: H_C \rightarrow H_C^\vee$  is an isomorphism.*

*Proof.*  $H_f$  is identically zero if and only if the symmetric bilinear forms  $F^t(h) \in S^2 H_C^\vee$  are degenerate for every  $h \in H_C$ . But this means that none of the maps  $\cdot h: H_C \rightarrow H_C^\vee$  is an isomorphism.

**COROLLARY 3.** *Let  $F \in S^3 H_C^\vee$  be a form whose associated map  $F^t: H_C \rightarrow S^2 H_C^\vee$  is not injective. Then we have  $H_f = 0$ .*

*Proof.* Let  $k \in \text{Ker}(F^t)$  be a non-zero element, and consider an arbitrary element  $h \in H_C$ . By definition of  $k$  we have  $F(k, h, v) = 0$  for all  $v \in H_C$ , i.e.  $k \cdot h \in H_C^\vee$  is zero.

REMARK 6. It is not difficult to show that  $F^t$  is not injective if and only if there exists a proper quotient  $H_C$  of  $H_C$ , and a form  $\bar{F} \in S^3 \bar{H}_C^\vee$  whose pull-back to  $H_C$  is the given form  $F$ . This means that the Hessians of cubic polynomials  $f \in \mathbf{C}[H_C]_3$  which ‘do not depend on all variables’ are automatically zero.

The converse holds for forms in  $b \leq 4$  variables, but not in general [G/N].

### 3.2 THE GIT QUOTIENT $S^3 H_C^\vee //_{SL(H_C)}$

Let  $V := S^3 H_C^\vee$  be the vector space of complex cubic forms. The reductive group  $G := SL(H_C)$  acts rationally on  $V$ , and therefore has a finitely generated ring  $\mathbf{C}[V]^G$  of invariants [H]. The inclusion  $\mathbf{C}[V]^G \subset \mathbf{C}[V]$  induces a regular map  $\pi: V \rightarrow V // G$  onto the affine variety  $V // G$  with coordinate ring  $\mathbf{C}[V]^G$ . It is well known that  $\pi$  is a categorical quotient, which is  $G$ -closed and  $G$ -separating, so that  $V // G$  parametrizes precisely the closed  $G$ -orbits in  $V$ . Recall that a point  $v \in V$  is semi-stable if  $o \notin \overline{G \cdot v}$ , and that  $v$  is stable if  $G \cdot v$  is closed in  $V$  and the isotropy group  $G_v$  is finite [M/F]. Denote the  $G$ -invariant, open subsets of semistable (stable) points in  $V$  by  $V^{ss}(V^s)$ .

The complement  $V \setminus V^{ss} = \pi^{-1}(\pi(0))$  consists of ‘Nullformen’, i.e. forms for which all polynomial invariants vanish. The open subset of stable points, which includes in particular all non-singular forms, has a geometric quotient, given by the restricted map  $\pi|_{V^s}: V^s \rightarrow \pi(V^s)$ .

REMARK 7. Let  $A_\circ \in GL(H)$  be a fixed automorphism of determinant  $\det A_\circ = -1$ , e.g.  $A_\circ = -id_H$  if  $b$  is odd.  $A_\circ$  induces a  $\mathbf{Z}_{/2}$ -action on  $S^3 H^\vee //_{SL(H)}$  and on  $S^3 H_C^\vee //_{SL(H_C)}$ , for which the map  $c$  is equivariant.

Let  $\hat{G} \subset GL(H_C)$  be the semi-direct product of  $SL(H_C)$  and  $\mathbf{Z}_{/2}$  generated by  $A_\circ$  and  $SL(H_C)$ . The invariant ring  $\mathbf{C}[V]^{\hat{G}}$  has an important topological interpretation: it consists of all polynomial invariants of complex cohomology rings of 1-connected, closed, oriented 6-dimensional manifolds with torsion-free homology.

#### EXAMPLE 5. Binary cubics ( $b = 2$ )

Choose linear coordinates  $X, Y$  on  $H_C$ , and write a cubic polynomial  $f \in \mathbf{C}[X, Y]_3$  in the form  $f = a_0 X^3 + 3a_1 X^2 Y + 3a_2 X Y^2 + a_3 Y^3$ .

We use  $a_0, a_1, a_2, a_3$  as coordinates on  $S^3 H_C^\vee$ , so that  $\mathbf{C}[S^3 H_C^\vee] = \mathbf{C}[a_0, a_1, a_2, a_3]$ . The discriminant  $\Delta(f)$  of  $f$  is a homogeneous