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## 3. ALGEBRA AND ARITHMETIC OF CUBIC FORMS

Let  $H$  be a finitely generated free  $\mathbf{Z}$ -module of rank  $b$ . In this section we want to study algebraic and arithmetic properties of symmetric trilinear forms  $F \in S^3 H^\vee$  on  $H$  which admit characteristic elements; ultimately we would like to describe the classification of those forms under the action of the general linear group  $GL(H)$ , i.e. we like to investigate (part of) the quotient  $S^3 H^\vee /_{GL(H)}$ .

From what we have said in sections 1 and 2, this is clearly equivalent to classifying the cohomology rings of 1-connected, closed, oriented, 6-dimensional manifolds without torsion, and with  $b_2 = b$ ,  $b_3 = 0$ . Furthermore, up to finite indeterminacy, this is also equivalent to classifying the homotopy types of these manifolds.

The proper setting for this arithmetic moduli problem can be found in C. Seshadri's paper [S]; here we investigate only its set-theoretic aspects. Let  $H_{\mathbf{C}} := H \otimes_{\mathbf{Z}} \mathbf{C}$  be the complexification of  $H$ , and let  $S^3 H_{\mathbf{C}}^\vee /_{SL(H_{\mathbf{C}})}$  be the quotient of the reductive group  $SL(H_{\mathbf{C}})$ . We obtain a natural map  $c: S^3 H^\vee /_{SL(H)} \rightarrow S^3 H_{\mathbf{C}}^\vee /_{SL(H_{\mathbf{C}})}$ , which allows us to break up the problem into three parts: the description of the quotient  $S^3 H_{\mathbf{C}}^\vee /_{SL(H_{\mathbf{C}})}$ , the investigation of the fibers of  $c$ , and the study of the remaining  $\mathbf{Z}/2$ -action on  $S^3 H^\vee /_{SL(H)}$  which is induced by the choice of an arbitrary automorphism  $A_0 \in GL(H)$  of determinant  $\det A_0 = -1$ .

## 3.1 ALGEBRAIC PROPERTIES OF CUBIC FORMS

Let  $H_{\mathbf{C}} = H \otimes_{\mathbf{Z}} \mathbf{C}$  be as above, and denote by  $\mathbf{C}[H_{\mathbf{C}}]_3$  the space of homogeneous polynomials of degree 3 on  $H_{\mathbf{C}}$ . There exists a linear polarization operator  $\text{Pol}: \mathbf{C}[H_{\mathbf{C}}]_3 \rightarrow S^3 H_{\mathbf{C}}^\vee$ , sending a homogeneous cubic polynomial  $f \in \mathbf{C}[H_{\mathbf{C}}]_3$  to the symmetric trilinear form  $F = \text{Pol}(f) \in S^3 H_{\mathbf{C}}^\vee$  which is related to  $f$  by the identity  $F(h, h, h) = 6f(h)$ . We will usually not distinguish between a cubic polynomial  $f$  and its associated form  $F = \text{Pol}(f)$ . On  $S^3 H_{\mathbf{C}}^\vee$  there exists a polynomial function  $\Delta: S^3 H_{\mathbf{C}}^\vee \rightarrow \mathbf{C}$ , the discriminant, which is homogeneous of degree  $b \cdot 2^{b-1}$ , and vanishes in a form  $F$  if and only if the associated cubic hypersurface  $(f)_0 \subset \mathbf{P}(H_{\mathbf{C}})$  has a singular point;  $\Delta$  is defined over  $\mathbf{Z}$  and is clearly invariant under the natural action of  $SL(H_{\mathbf{C}})$ .

REMARK 4. Of course, a discriminant function  $\Delta$  exists for forms of arbitrary degree  $d$ ; in the general case  $\Delta$  is homogeneous of degree  $b \cdot (d-1)^{b-1}$  on  $S^d H_{\mathbf{C}}^\vee$ .

PROPOSITION 5. *Fix a symmetric trilinear form  $F \in S^3 H_C^\vee$  and an element  $h \in H_C \setminus \{0\}$  with  $f(h) = 0$ . The associated point  $\langle h \rangle \in \mathbf{P}(H_C)$  is a singular point of the cubic hypersurface  $(f)_\circ \subset \mathbf{P}(H_C)$  if and only if the linear form  $h^2 \in H_C^\vee$  is zero. The existence of at least one such point is equivalent to the vanishing of the discriminant.*

*Proof.* From  $f(h + tv) = f(h) + 3th^2 \cdot v + 3t^2 h \cdot v^2 + t^3 v^3$  for every  $v \in H_C, t \in \mathbf{C}$  we find  $\frac{d}{dt} \big|_0 f(h + tv) = 3h^2 \cdot v$ , i.e.  $h^2 \in H_C^\vee$  defines the differential of  $f$  in  $h$ .

REMARK 5.  $\mathbf{Q}$ -rational points in  $(f)_\circ \subset \mathbf{P}(H_C)$ , and  $\mathbf{Q}$ -rational singularities of  $(f)_\circ$  have geometric significance if the cubic  $f$  is defined by the cup-form of a 6-manifold  $X$ . In fact, integral classes  $h \in H^2(X, \mathbf{Z})$  correspond to homotopy classes of maps to  $\mathbf{P}_\mathbf{C}^3$ ; such a map factors over  $\mathbf{P}_\mathbf{C}^2 \subset \mathbf{P}_\mathbf{C}^3$  if and only if  $h^3 = 0$ ; if it factors over  $\mathbf{P}_\mathbf{C}^1 \subset \mathbf{P}_\mathbf{C}^3$ , then clearly  $h^2 = 0$ . The converse will probably not always be true since, in general, the cohomology ring does not determine the homotopy type.

In addition to the invariant discriminant  $\Delta(f)$  of a polynomial  $f$ , we will also need a fundamental covariant  $H_f$ , the Hessian of  $f$ . Let  $F = \text{Pol}(f) \in S^3 H_C^\vee$  be the polarization of  $f \in \mathbf{C}[H_C]_3$ ; the Hessian of  $f$  can then be defined as the composition  $H_f: H_C \xrightarrow{F^t} S^2 H_C^\vee \xrightarrow{\text{disc}} \mathbf{C}$ , i.e.  $H_f$  is the homogeneous polynomial function of degree  $b$  on  $H_C$  given by  $H_f(h) = \text{disc}(F^t(h))$ . In terms of linear coordinates  $\xi_1, \dots, \xi_b$  on  $H$  one finds the more familiar expression  $H_f = \det \left( \frac{\partial^2}{\partial \xi_i \partial \xi_j} f \right)$ .

PROPOSITION 6. *Let  $F \in S^3 H_C^\vee$  be a symmetric trilinear form. The Hessian of  $F$  is identically zero if and only if there exists no element  $h \in H_C$  for which the map  $\cdot h: H_C \rightarrow H_C^\vee$  is an isomorphism.*

*Proof.*  $H_f$  is identically zero if and only if the symmetric bilinear forms  $F^t(h) \in S^2 H_C^\vee$  are degenerate for every  $h \in H_C$ . But this means that none of the maps  $\cdot h: H_C \rightarrow H_C^\vee$  is an isomorphism.

COROLLARY 3. *Let  $F \in S^3 H_C^\vee$  be a form whose associated map  $F^t: H_C \rightarrow S^2 H_C^\vee$  is not injective. Then we have  $H_f = 0$ .*

*Proof.* Let  $k \in \text{Ker}(F^t)$  be a non-zero element, and consider an arbitrary element  $h \in H_C$ . By definition of  $k$  we have  $F(k, h, v) = 0$  for all  $v \in H_C$ , i.e.  $k \cdot h \in H_C^\vee$  is zero.

REMARK 6. It is not difficult to show that  $F^t$  is not injective if and only if there exists a proper quotient  $\bar{H}_C$  of  $H_C$ , and a form  $\bar{F} \in S^3 \bar{H}_C^\vee$  whose pull-back to  $H_C$  is the given form  $F$ . This means that the Hessians of cubic polynomials  $f \in \mathbf{C}[H_C]_3$  which ‘do not depend on all variables’ are automatically zero.

The converse holds for forms in  $b \leq 4$  variables, but not in general  $[G/N]$ .

### 3.2 THE GIT QUOTIENT $S^3 H_C^\vee //_{SL(H_C)}$

Let  $V := S^3 H_C^\vee$  be the vector space of complex cubic forms. The reductive group  $G := SL(H_C)$  acts rationally on  $V$ , and therefore has a finitely generated ring  $\mathbf{C}[V]^G$  of invariants  $[H]$ . The inclusion  $\mathbf{C}[V]^G \subset \mathbf{C}[V]$  induces a regular map  $\pi: V \rightarrow V//_G$  onto the affine variety  $V//_G$  with coordinate ring  $\mathbf{C}[V]^G$ . It is well known that  $\pi$  is a categorical quotient, which is  $G$ -closed and  $G$ -separating, so that  $V//_G$  parametrizes precisely the closed  $G$ -orbits in  $V$ . Recall that a point  $v \in V$  is semi-stable if  $0 \notin \bar{G} \cdot v$ , and that  $v$  is stable if  $G \cdot v$  is closed in  $V$  and the isotropy group  $G_v$  is finite  $[M/F]$ . Denote the  $G$ -invariant, open subsets of semistable (stable) points in  $V$  by  $V^{ss}(V^s)$ .

The complement  $V \setminus V^{ss} = \pi^{-1}(\pi(0))$  consists of ‘Nullformen’, i.e. forms for which all polynomial invariants vanish. The open subset of stable points, which includes in particular all non-singular forms, has a geometric quotient, given by the restricted map  $\pi|_{V^s}: V^s \rightarrow \pi(V^s)$ .

REMARK 7. Let  $A_0 \in GL(H)$  be a fixed automorphism of determinant  $\det A_0 = -1$ , e.g.  $A_0 = -id_H$  if  $b$  is odd.  $A_0$  induces a  $\mathbf{Z}_{/2}$ -action on  $S^3 H^\vee /_{SL(H)}$  and on  $S^3 H_C^\vee /_{SL(H_C)}$ , for which the map  $c$  is equivariant.

Let  $\hat{G} \subset GL(H_C)$  be the semi-direct product of  $SL(H_C)$  and  $\mathbf{Z}_{/2}$  generated by  $A_0$  and  $SL(H_C)$ . The invariant ring  $\mathbf{C}[V]^{\hat{G}}$  has an important topological interpretation: it consists of all polynomial invariants of complex cohomology rings of 1-connected, closed, oriented 6-dimensional manifolds with torsion-free homology.

#### EXAMPLE 5. Binary cubics ( $b = 2$ )

Choose linear coordinates  $X, Y$  on  $H_C$ , and write a cubic polynomial  $f \in \mathbf{C}[X, Y]_3$  in the form  $f = a_0 X^3 + 3a_1 X^2 Y + 3a_2 X Y^2 + a_3 Y^3$ .

We use  $a_0, a_1, a_2, a_3$  as coordinates on  $S^3 H_C^\vee$ , so that  $\mathbf{C}[S^3 H_C^\vee] = \mathbf{C}[a_0, a_1, a_2, a_3]$ . The discriminant  $\Delta(f)$  of  $f$  is a homogeneous