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Autor: Okonek, Ch. / Van de Ven, A.
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3. ALGEBRA AND ARITHMETIC OF CUBIC FORMS

Let H be a finitely generated free $\mathbf{Z}$-module of rank b. In this section we want to study algebraic and arithmetic properties of symmetric trilinear forms $F \in S^3H^\vee$ on H which admit characteristic elements; ultimately we would like to describe the classification of those forms under the action of the general linear group $GL(H)$, i.e. we like to investigate (part of) the quotient $S^3H^\vee/_{GL(H)}$.

From what we have said in sections 1 and 2, this is clearly equivalent to classifying the cohomology rings of 1-connected, closed, oriented, 6-dimensional manifolds without torsion, and with $b_2 = b$, $b_3 = 0$. Furthermore, up to finite indeterminancy, this is also equivalent to classifying the homotopy types of these manifolds.

The proper setting for this arithmetic moduli problem can be found in C. Seshadri's paper [S]; here we investigate only its set-theoretic aspects. Let $H_{\mathbf{C}} := H \otimes_{\mathbf{Z}} \mathbf{C}$ be the complexification of H, and let $S^3H_{\mathbf{C}}^\vee/_{SL(H_{\mathbf{C}})}$ be the quotient of the reductive group $SL(H_{\mathbf{C}})$. We obtain a natural map $c: S^3H^\vee/_{SL(H)} \to S^3H_{\mathbf{C}}^\vee/_{SL(H_{\mathbf{C}})}$, which allows us to break up the problem into three parts: the description of the quotient $S^3H_{\mathbf{C}}^\vee/_{SL(H_{\mathbf{C}})}$, the investigation of the fibers of c, and the study of the remaining $\mathbf{Z}_{/2}$-action on $S^3H^\vee/_{SL(H)}$ which is induced by the choice of an arbitrary automorphism $A_\circ \in GL(H)$ of determinant $\det A_\circ = -1$.

3.1 ALGEBRAIC PROPERTIES OF CUBIC FORMS

Let $H_{\mathbf{C}} = H \otimes_{\mathbf{Z}} \mathbf{C}$ be as above, and denote by $\mathbf{C}[H_{\mathbf{C}}]_3$ the space of homogeneous polynomials of degree 3 on $H_{\mathbf{C}}$. There exists a linear polarization operator $\mathrm{Pol}: \mathbf{C}[H_{\mathbf{C}}]_3 \to S^3H_{\mathbf{C}}^\vee$, sending a homogeneous cubic polynomial $f \in \mathbf{C}[H_{\mathbf{C}}]_3$ to the symmetric trilinear form $F = \mathrm{Pol}(f) \in S^3H_{\mathbf{C}}^\vee$ which is related to f by the identity $F(h, h, h) = 6f(h)$. We will usually not distinguish between a cubic polynomial f and its associated form $F = \mathrm{Pol}(f)$. On $S^3H_{\mathbf{C}}^\vee$ there exists a polynomial function $\Delta: S^3H_{\mathbf{C}}^\vee \to \mathbf{C}$, the discriminant, which is homogeneous of degree $b \cdot 2^{b-1}$, and vanishes in a form F if and only if the associated cubic hypersurface $(f)_\circ \subset \mathbf{P}(H_{\mathbf{C}})$ has a singular point; Δ is defined over $\mathbf{Z}$ and is clearly invariant under the natural action of $SL(H_{\mathbf{C}})$.

REMARK 4. Of course, a discriminant function Δ exists for forms of arbitrary degree d; in the general case Δ is homogeneous of degree $b \cdot (d-1)^{b-1}$ on $S^dH_{\mathbf{C}}^\vee$.

PROPOSITION 5. *Fix a symmetric trilinear form* $F \in S^3 H_{\mathbf{C}}^{\vee}$ *and an element* $h \in H_{\mathbf{C}} \setminus \{0\}$ *with* $f(h) = 0$. *The associated point* $\langle h \rangle \in \mathbf{P}(H_{\mathbf{C}})$ *is a singular point of the cubic hypersurface* $(f)_0 \subset \mathbf{P}(H_{\mathbf{C}})$ *if and only if the linear form* $h^2 \in H_{\mathbf{C}}^{\vee}$ *is zero. The existence of at least one such point is equivalent to the vanishing of the discriminant.*

Proof. From $f(h + tv) = f(h) + 3th^2 \cdot v + 3t^2 h \cdot v^2 + t^3 v^3$ for every $v \in H_{\mathbf{C}}, t \in \mathbf{C}$ we find $\frac{d}{dt}|_0 f(h + tv) = 3h^2 \cdot v$, i.e. $h^2 \in H_{\mathbf{C}}^{\vee}$ defines the differential of f in h.

REMARK 5. $\mathbf{Q}$-rational points in $(f)_0 \subset \mathbf{P}(H_{\mathbf{C}})$, and $\mathbf{Q}$-rational singularities of $(f)_0$ have geometric significance if the cubic f is defined by the cup-form of a 6-manifold X. In fact, integral classes $h \in H^2(X, \mathbf{Z})$ correspond to homotopy classes of maps to $\mathbf{P}_{\mathbf{C}}^3$; such a map factors over $\mathbf{P}_{\mathbf{C}}^2 \subset \mathbf{P}_{\mathbf{C}}^3$ if and only if $h^3 = 0$; if it factors over $\mathbf{P}_{\mathbf{C}}^1 \subset \mathbf{P}_{\mathbf{C}}^3$, then clearly $h^2 = 0$. The converse will probably not always be true since, in general, the cohomology ring does not determine the homotopy type.

In addition to the invariant discriminant $\Delta(f)$ of a polynomial f, we will also need a fundamental covariant H_f, the Hessian of f. Let $F = \mathrm{Pol}(f) \in S^3 H_{\mathbf{C}}^{\vee}$ be the polarization of $f \in \mathbf{C}[H_{\mathbf{C}}]_3$; the Hessian of f can then be defined as the composition $H_f : H_{\mathbf{C}} \xrightarrow{F^t} S^2 H_{\mathbf{C}}^{\vee} \xrightarrow{\mathrm{disc}} \mathbf{C}$, i.e. H_f is the homogeneous polynomial function of degree b on $H_{\mathbf{C}}$ given by $H_f(h) = \mathrm{disc}(F^t(h))$. In terms of linear coordinates $\xi_1, \cdots, \xi_b$ on H one finds the more familiar expression $H_f = \det\left(\frac{\partial^2}{\partial\xi_i \partial\xi_j} f\right)$.

PROPOSITION 6. *Let* $F \in S^3 H_{\mathbf{C}}^{\vee}$ *be a symmetric trilinear form. The Hessian of* F *is identically zero if and only if there exists no element* $h \in H_{\mathbf{C}}$ *for which the map* $\cdot h : H_{\mathbf{C}} \to H_{\mathbf{C}}^{\vee}$ *is an isomorphism.*

Proof. H_f is identically zero if and only if the symmetric bilinear forms $F^t(h) \in S^2 H_{\mathbf{C}}^{\vee}$ are degenerate for every $h \in H_{\mathbf{C}}$. But this means that none of the maps $\cdot h : H_{\mathbf{C}} \to H_{\mathbf{C}}^{\vee}$ is an isomorphism.

COROLLARY 3. *Let* $F \in S^3 H_{\mathbf{C}}^{\vee}$ *be a form whose associated map* $F^t : H_{\mathbf{C}} \to S^2 H_{\mathbf{C}}^{\vee}$ *is not injective. Then we have* $H_f = 0$.

Proof. Let $k \in \mathrm{Ker}(F^t)$ be a non-zero element, and consider an arbitrary element $h \in H_{\mathbf{C}}$. By definition of k we have $F(k, h, v) = 0$ for all $v \in H_{\mathbf{C}}$, i.e. $k \cdot h \in H_{\mathbf{C}}^{\vee}$ is zero.

REMARK 6. It is not difficult to show that F^t is not injective if and only if there exists a proper quotient $\bar{H}_{\mathbf{C}}$ of $H_{\mathbf{C}}$, and a form $\bar{F} \in S^3\bar{H}_{\mathbf{C}}^{\vee}$ whose pull-back to $H_{\mathbf{C}}$ is the given form F. This means that the Hessians of cubic polynomials $f \in \mathbf{C}[H_{\mathbf{C}}]_3$ which 'do not depend on all variables' are automatically zero.

The converse holds for forms in $b \leqslant 4$ variables, but not in general [G/N].

3.2 THE GIT QUOTIENT $S^3H_{\mathbf{C}}^{\vee}//_{SL(H_{\mathbf{C}})}$

Let $V := S^3H_{\mathbf{C}}^{\vee}$ be the vector space of complex cubic forms. The reductive group $G := SL(H_{\mathbf{C}})$ acts rationally on V, and therefore has a finitely generated ring $\mathbf{C}[V]^G$ of invariants [H]. The inclusion $\mathbf{C}[V]^G \subset \mathbf{C}[V]$ induces a regular map $\pi: V \to V//_G$ onto the affine variety $V//_G$ with coordinate ring $\mathbf{C}[V]^G$. It is well known that π is a categorical quotient, which is G-closed and G-separating, so that $V//_G$ parametrizes precisely the closed G-orbits in V. Recall that a point $v \in V$ is semi-stable if $o \notin \overline{G \cdot v}$, and that v is stable if $G \cdot v$ is closed in V and the isotropy group G_v is finite [M/F]. Denote the G-invariant, open subsets of semistable (stable) points in V by $V^{ss}(V^s)$.

The complement $V \setminus V^{ss} = \pi^{-1}(\pi(0))$ consists of 'Nullformen', i.e. forms for which all polynomial invariants vanish. The open subset of stable points, which includes in particular all non-singular forms, has a geometric quotient, given by the restricted map $\pi \mid V^s: V^s \to \pi(V^s)$.

REMARK 7. Let $A_{\circ} \in GL(H)$ be a fixed automorphism of determinant $\det A_{\circ} = -1$, e.g. $A_{\circ} = -id_H$ if b is odd. $A_{\circ}$ induces a $\mathbf{Z}_{/2}$-action on $S^3H^{\vee}/_{SL(H)}$ and on $S^3H_{\mathbf{C}}^{\vee}/_{SL(H_{\mathbf{C}})}$, for which the map c is equivariant.

Let $\hat{G} \subset GL(H_{\mathbf{C}})$ be the semi-direct product of $SL(H_{\mathbf{C}})$ and $\mathbf{Z}_{/2}$ generated by $A_{\circ}$ and $SL(H_{\mathbf{C}})$. The invariant ring $\mathbf{C}[V]^{\hat{G}}$ has an important topological interpretation: it consists of all polynomial invariants of complex cohomology rings of 1-connected, closed, oriented 6-dimensional manifolds with torsion-free homology.

EXAMPLE 5. Binary cubics ($b = 2$)

Choose linear coordinates X, Y on $H_{\mathbf{C}}$, and write a cubic polynomial $f \in \mathbf{C}[X, Y]_3$ in the form $f = a_0X^3 + 3a_1X^2Y + 3a_2XY^2 + a_3Y^3$.

We use a_0, a_1, a_2, a_3 as coordinates on $S^3H_{\mathbf{C}}^{\vee}$, so that $\mathbf{C}[S^3H_{\mathbf{C}}^{\vee}] = \mathbf{C}[a_0, a_1, a_2, a_3]$. The discriminant $\Delta(f)$ of f is a homogeneous