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*Proof.*  $W_\circ$  is characteristic for  $F$  if and only if  $q_{\bar{F}} = \bar{F}^t(W_\circ)$ .

In terms of a  $\mathbf{Z}$ -basis  $\{e_1, \dots, e_b\}$  for  $H$  the condition  $q_{\bar{F}} \in \text{Im}(\bar{F}^t)$  translates into a simple rank condition over  $\mathbf{Z}_{/2}$ : the  $\mathbf{Z}_{/2}$ -rank of the  $b \times \binom{b+1}{2}$ -matrix  $A$  representing  $\bar{F}^t$  must be equal to the  $\mathbf{Z}_{/2}$ -rank of the matrix  $A$  extended by the column  $(\bar{e}_i \cdot \bar{e}_j \cdot (\bar{e}_i + \bar{e}_j))_{1 \leq i \leq j \leq b}$

EXAMPLE 3. Let  $H = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$  be free of rank 2,  $F \in S^3 H^\vee$  given by  $e_1^3 = a$ ,  $e_1^2 e_2 = b$ ,  $e_1 e_2^2 = c$ ,  $e_2^3 = d$  with  $a, b, c, d \in \mathbf{Z}$ . The rank condition becomes

$$rk_2 \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \\ \bar{b} & \bar{c} \end{bmatrix} = rk_2 \begin{bmatrix} \bar{a} & \bar{b} & \bar{0} \\ \bar{c} & \bar{d} & \bar{0} \\ \bar{b} & \bar{c} & \bar{b+c} \end{bmatrix}$$

## 2.2 HOMOTOPY TYPES WITH A GIVEN COHOMOLOGY RING

Our next task is to describe the set of oriented homotopy types of 1-connected, closed, oriented, 6-dimensional manifolds with a fixed torsion-free cohomology ring.

From Žubr's classification theorem we know that in algebraic terms this means the following: fix a non-negative integer  $r_\circ$ , a finitely generated free abelian group  $H_\circ$ , and a symmetric trilinear form  $F_\circ \in S^3 H_\circ^\vee$  which admits characteristic elements.

Let  $\mathcal{M}(r_\circ, H_\circ, F_\circ)$  be the set of 1-connected, closed, oriented, 6-dimensional manifolds  $X$  with  $b_3(X) = 2r_\circ$ , such that there exists an isomorphism  $\alpha: H_\circ \rightarrow H^2(X, \mathbf{Z})$  with  $\alpha^* F_X = F_\circ$ . Denote by  $\text{Aut}(F_\circ)$  the subgroup of  $\mathbf{Z}$ -automorphisms of  $H_\circ$  which leave  $F_\circ \in S^3 H_\circ^\vee$  invariant;  $\text{Aut}(F_\circ)$  acts on pairs  $(w, [l]) \in \bar{H}_\circ \times H_\circ^\vee /_{48 H_\circ^\vee} /_{U_{F_\circ}}$  in a natural way:

$$\gamma \cdot (w, [l]) := (\gamma(w), (\gamma^{-1})^* [l]).$$

Let  ${}_{\text{Aut}(F_\circ)} \backslash \bar{H}_\circ \times H_\circ^\vee /_{48 H_\circ^\vee} /_{U_{F_\circ}}$  be the set of  $\text{Aut}(F_\circ)$ -orbits.

A manifold  $X$  in  $\mathcal{M}(r_\circ, H_\circ, F_\circ)$  and an isomorphism  $\alpha: H_\circ \rightarrow H^2(X, \mathbf{Z})$  with  $\alpha^* F_X = F_\circ$  yields a well-defined  $\text{Aut}(F_\circ)$ -orbit:

$$(\alpha^{-1}(w_2(X)), \alpha^* [p_1(X) + 24T]) \pmod{\text{Aut}(F_\circ)},$$

where  $T \in H^4(X, \mathbf{Z})$  is an arbitrary integral lifting of  $\tau(X) \in H^4(X, \mathbf{Z}_{/2})$ .

The set of oriented homotopy types  $\mathcal{M}(r_\circ, H_\circ, F_\circ)/_{\sim}$  of manifolds in  $\mathcal{M}(r_\circ, H_\circ, F_\circ)$  can now be described in the following way:

**PROPOSITION 3.** *The assignment  $X \mapsto (\alpha^{-1}(w_2(X)), \alpha^*[p_1(X) + 24T])$  (modulo  $\text{Aut}(F_\circ)$ ) defines an injection.*

$$I: \mathcal{M}(r_\circ, H_\circ, F_\circ)/_ \simeq \rightarrow {}_{\text{Aut}(F_\circ)} \backslash \bar{H}_\circ \times H_\circ^\vee /_{48H_\circ^\vee} /_{U_{F_\circ}}.$$

*Proof.* Suppose  $X$  and  $X'$  are manifolds in  $\mathcal{M}(r_\circ, H_\circ, F_\circ)$ ,  $\alpha: H_\circ \rightarrow H^2(X, \mathbf{Z})$  and  $\alpha': H_\circ \rightarrow H^2(X', \mathbf{Z})$  isomorphisms with  $\alpha^*F_X = F_\circ$  and  $(\alpha')^*F_{X'} = F_\circ$ .  $X$  and  $X'$  have the same image under  $I$  iff there exists an automorphism  $\gamma \in \text{Aut}(F_\circ)$  with  $\gamma\alpha^{-1}(w_2(X)) = (\alpha')^{-1}w_2(X')$  and  $(\gamma^{-1})^*\alpha^*[p_1(X) + 24T] = (\alpha')^*[p_1(X') + 24T']$ . Consider  $\beta := \alpha \circ \gamma \circ \alpha^{-1}: H^2(X, \mathbf{Z}) \rightarrow H^2(X', \mathbf{Z})$ ;  $\beta$  is obviously an isomorphism with  $\beta^*F_{X'} = F_X$ ,  $\beta w_2(X) = w_2(X')$ , and  $\beta^*[p_1(X') + 24T'] = [p_1(X) + 24T]$ ; but this means that the systems of invariants associated with  $X$  and  $X'$  are weakly equivalent, and therefore  $X$  and  $X'$  oriented homotopy equivalent.

A complete description of the set  $\mathcal{M}(r_\circ, H_\circ, F_\circ)/_ \simeq$  i.e. of the image of  $I$  is only possible if the automorphism group  $\text{Aut}(F_\circ)$  is known; this can be a serious problem, but we will see that the ‘general’ automorphism group is finite (and usually small), so that the next proposition gives a reasonable estimate for the number of elements in  $\mathcal{M}(r_\circ, H_\circ, F_\circ)/_ \simeq$ .

**PROPOSITION 4.** *Fix  $r_\circ \in \mathbf{N}$ , a finitely generated free abelian group  $H_\circ$ , and a symmetric trilinear form  $F_\circ \in S^3 H_\circ^\vee$  which admits characteristic elements. Set  $b := rk_{\mathbf{Z}} H_\circ$ ,  $s := rk_{\mathbf{Z}/2}(\bar{F}_\circ^t)$ , and let  $t := rk_{\mathbf{Z}/2}(\cdot_{\bar{F}_\circ})$  be the  $\mathbf{Z}/2$ -rank of the  $\mathbf{Z}/2$ -linear square map  $\cdot_{\bar{F}_\circ}: \bar{H}_\circ \rightarrow \bar{H}_\circ^\vee$  sending  $\bar{u} \in \bar{H}_\circ$  to  $\bar{u}^2 \in \bar{H}_\circ^\vee$ . Then  $\mathcal{M}(r_\circ, H_\circ, F_\circ)/_ \simeq$  contains at most  $2^{2b-s-t}$  elements.*

*Proof.* Fix any admissible system of invariants  $(r_\circ, H_\circ, w_\circ, \tau_\circ, F_\circ, p_\circ)$  for a manifold in  $\mathcal{M}(r_\circ, H_\circ, F_\circ)$ . Given  $(r_\circ, H_\circ, F_\circ)$ , we know from the last lemma that the possible elements  $w_\circ$  form a coset of  $\text{Ker}(\bar{F}_\circ^t)$  in  $\bar{H}_\circ$ , so that there exist precisely  $2^{b-s}$  such elements. It remains to count the classes  $[l] \in H_\circ^\vee /_{48H_\circ^\vee} /_{U_{F_\circ}}$ , such that the  $\text{Aut}(F_\circ)$ -orbit of  $(w_\circ, [p_\circ + 24T_\circ + l])$  lies in the image of  $I$ .

To understand the latter condition we fix integral liftings  $W_\circ \in H_\circ$ ,  $T_\circ \in H_\circ^\vee$  of  $w_\circ$  and  $\tau_\circ$  satisfying the admissibility conditions

- i)  $W_\circ^3 \equiv (p_\circ + 24T_\circ)(W_\circ) \pmod{48}$
- ii)  $p_\circ(x) \equiv 4x^3 + 6x^2 W_\circ + 3x W_\circ^2 \pmod{24} \quad \forall x \in H_\circ$ .

Clearly the  $\text{Aut}(F_\circ)$ -orbit of  $(w_\circ, [p_\circ + 24T_\circ + l])$  lies in the image of  $I$  if and only if

- i')  $W_\circ^3 \equiv (p_\circ + 24T_\circ + l)(W_\circ) \pmod{48}$ ,  
ii')  $(p_\circ + l)(x) \equiv 4x^3 + 6x^2 W_\circ + 3x W_\circ^2 \pmod{24} \quad \forall x \in H_\circ$ ,  
which is equivalent to  $l(W_\circ) \equiv 0 \pmod{48}$ , and  $l \equiv 0 \pmod{24H_\circ^\vee}$  because of i) and ii).

Now, by definition of the subgroup  $U_{F_\circ} \subset H_\circ^\vee /_{48H_\circ^\vee}$  we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
& \text{Ker}(\cdot_{\bar{F}_\circ}) & & 0 & & & \\
& \downarrow & & \downarrow & & & \\
0 \rightarrow \text{Ker}(24 \cdot_{\bar{F}_\circ}) & \hookrightarrow & H_\circ /_{2H_\circ} & \xrightarrow{24 \cdot_{\bar{F}_\circ}} & U_{F_\circ} & \rightarrow & 0 \\
& & \cdot_{\bar{F}_\circ} \downarrow & & \downarrow & & \\
0 \rightarrow & H_\circ^\vee /_{2H_\circ^\vee} & \xrightarrow{24} & H_\circ^\vee /_{48H_\circ^\vee} & \rightarrow & H_\circ^\vee /_{24H_\circ^\vee} \rightarrow 0 \\
& \downarrow & & \downarrow & & \parallel & \\
0 \rightarrow & \text{Coker}(\cdot_{\bar{F}_\circ}) & \rightarrow & H_\circ^\vee /_{48H_\circ^\vee} /_{U_{F_\circ}} & \rightarrow & H_\circ^\vee /_{24H_\circ^\vee} \rightarrow 0 \\
& \downarrow & & \downarrow & & & \\
& 0 & & 0 & & &
\end{array}$$

The number of elements  $[l] \in H_\circ^\vee /_{48H_\circ^\vee} /_{U_{F_\circ}}$  to be counted coincides therefore with the cardinality of the kernel of the map  $ev(w_\circ) : \text{Coker}(\cdot_{\bar{F}_\circ}) \rightarrow \mathbf{Z}_{/2}$  induced by evaluation in  $w_\circ$ . This number is at most  $2^{b-t}(2^{b-t-1}$  if  $w_\circ \neq 0$  and  $t \neq b$ ).

**COROLLARY 2.** *If the  $\mathbf{Z}_{/2}$ -rank  $s = rk_{\mathbf{Z}_{/2}}(\cdot_{\bar{F}_\circ})$  is maximal, then  $\mathcal{M}(r_\circ, H_\circ, F_\circ) / \sim$  contains at most one class.*

*Proof.* Suppose  $\cdot_{\bar{F}_\circ} : \bar{H}_\circ \rightarrow \bar{H}_\circ^\vee$  is surjective; then  $\bar{F}_\circ^t : \bar{H}_\circ \rightarrow S^2 \bar{H}_\circ^\vee$  must have a trivial kernel, since  $\bar{h}\bar{x}^2 = 0$  for all  $\bar{x} \in \bar{H}_\circ$  implies  $\bar{h} = 0$  if every linear form is a square. But this means  $s = t = b$ , so that  $\mathcal{M}(r_\circ, H_\circ, F_\circ) / \sim$  has at most one element.

**EXAMPLE 4.** Let  $H_\circ = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$ ,  $e_1^3 = a$ ,  $e_1^2 e_2 = b$ ,  $e_1 e_2^2 = c$ ,  $e_2^3 = d$ . If  $\bar{b} \equiv \bar{c} \pmod{2}$ , and  $\bar{a}\bar{d} - \bar{b}\bar{c} \equiv 1 \pmod{2}$ , then  $\mathcal{M}(r_\circ, H_\circ, F_\circ) / \sim$  contains precisely one class for every  $r_\circ \geq 0$ .